ANALYTIC GEOMETRY

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ELEMENTS

OF

ANALYTIC GEOMETRY

BY

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NOTE TO THE SECOND EDITION.

In this edition such changes have been made as actual experience in the class-room has shown to be desirable.

A chapter on Higher Plane Curves, and four chapters on Solid Geometry have been added, making the work sufficiently extensive for our best schools and colleges.

An effort has been made to have this edition free from errors. It is not likely, however, that this effort has been entirely successful, and the author will be very grateful to any reader who will notify him of any needed corrections.

NOTE TO THE EDITION OF 1898.

The old plates have become so worn that it is necessary to have new plates. A few verbal changes have been made. No changes, however, have been made that will prevent the using of old and new books together.

March, 1898. G. A WENTWORTH.

PREFACE.

THIS book is intended for beginners. As beginners generally find great difficulty in comprehending the connection between a locus and its equation, the opening chapter is devoted mainly to an attempt, by means of easy illustrations and examples, to make this connection clear.

Each chapter abounds in exercises; for it is only by solving problems which require some degree of original thought that any real mastery of the study can be gained.

The more difficult propositions have been put at the ends of the chapters, under the heading of "Supplementary Propositions." This arrangement makes it possible for every teacher to mark out his own course. The simplest course will be Chapters I.–III. and Chapters V.–VII., with Review Exercises and Supplementary Propositions left out. Between this course and the entire work the teacher can exercise his choice, and take just so much as time and circumstances will allow.

The author has gathered his materials from many sources, but he is particularly indebted to the English treatise of CHARLES SMITH. Special acknowledgment is due to G. A. HILL, A.M., of Cambridge, Mass., and to Prof. J. M. TAYLOR, Colgate University, Hamilton, New York, for assistance in the preparation of the book.

Corrections and suggestions will be thankfully received.

G. A. WENTWORTH.

EXETER, N.H., January, 1888.

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ANALYTIC GEOMETRY.

PART I.—PLANE GEOMETRY.

CHAPTER I.

LOCI AND THEIR EQUATIONS.

Rectilinear System of Coördinates.

1. Let XX' and YY' (Fig. 1) be two fixed lines intersecting in the point O. These lines divide the plane in which they lie into four portions.

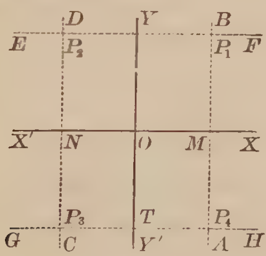


Fig. 1.

Let these parts be called **Quadrants** (as in Trigonometry), and distinguished by naming the area between OX and OY the *first* quadrant; that between OY and OX' the *second* quadrant; that between OX' and OY' the *third* quadrant; and that between OY' and OX the *fourth* quadrant.

Suppose the position of a point is described by saying that its distance from YY', expressed in terms of some chosen unit of length, is 3, and its distance from XX' is 4, it being understood that the distance from either line is measured parallel to the other. It is clear that in each quadrant there is one point, and only one, that will satisfy these conditions. The position of the point in each quadrant may be found by drawing parallels to YY' at the distance 3 from YY', and parallels to XX' at the distance 4 from XX'; then the intersections P_1, P_2, P_3, P_4 satisfy the given conditions.

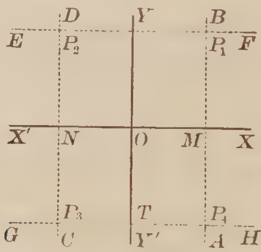


Fig. 1.

2. In order to determine which one of the four points, P_1, P_2, P_3, P_4, is meant, we adopt the rule that *opposite directions shall be indicated by unlike signs.* As in Trigonometry, distances measured from YY' *to the right* are considered *positive; to the left, negative.* Distances measured from XX' *upward* are *positive: downward, negative.* Then the position of P_1 will be denoted by $+3, +4$; of P_2, by $-3, +4$; of P_3, by $-3. -4$; of P_4, by $+3, -4$.

3. The fixed lines XX' and YY' are called the **Axes of Coördinates**; XX' is called the **Axis of Abscissas,** or **Axis of x**; YY', the **Axis of Ordinates,** or **Axis of y.** The intersection O is called the **Origin**.

The two distances (with signs prefixed) that determine the position of a point are called the **Coördinates** of the point; the distance of the point from YY' is called its **Abscissa**; and the distance from XX', its **Ordinate.**

Abscissas are usually denoted by x, and ordinates by y. A point is represented algebraically by simply writing the values of its coördinates within a parenthesis, that of the abscissa being always written first.

Thus P_1 (Fig. 1) is the point $(3, 4)$; P_2, the point $(-3, 4)$; P_3, the point $(-3, -4)$; P_4, the point $(3, -4)$. In general, the point whose coördinates are x and y is the point (x, y).

4. This method of determining the position of a point in a plane is called the **Rectilinear System of Coördinates.** The coördinates are called *rectangular* or *oblique*, according as the axes are rectangular or oblique; that is, according as the axes intersect at right or oblique angles. In the first three chapters we shall use only rectangular coördinates.

NOTE. The first man to employ this method successfully in investigating the properties of certain figures was the French philosopher Descartes, whose work on Geometry appeared in the year 1637.

Exercise 1.

1. What are the coördinates of the origin?

2. In what quadrants are the following points (a and b being given lengths):

$$(-a, -b),\quad (-a, b),\quad (a, b),\quad (a, -b)\ ?$$

3. To what quadrants is a point limited if its abscissa is positive? negative? ordinate positive? ordinate negative?

4. In what line does a point lie if its abscissa $= 0$? if its ordinate $= 0$?

5. A point (x, y) moves parallel to the axis of x; which one of its coördinates remains constant in value?

6. Construct or *plot* the points: $(2, 3)$, $(3, -3)$, $(-1, -3)$, $(-4, 4)$, $(3, 0)$, $(-3, 0)$, $(0, 4)$, $(0, -1)$, $(0, 0)$.

NOTE. To *plot* a point is to mark its proper position on paper, when its coördinates are given. The first thing to do is to draw the two axes. The rest of the work is obvious after a study of Nos. 1–3.

7. Construct the triangle whose vertices are the points $(2, 4)$, $(-2, 7)$, $(-6, -8)$.

8. Construct the quadrilateral whose vertices are the points $(7, 2)$, $(0, -9)$, $(-7, -1)$, $(-6, 4)$.

9. Construct the quadrilateral whose vertices are $(-3, 6)$, $(-3, 0)$, $(3, 0)$, $(3, 6)$. What kind of quadrilateral is it?

10. Mark the four points $(2, 1)$, $(4, 3)$, $(2, 5)$, and $(0, 3)$, and connect them by straight lines. What kind of a figure do these four lines enclose?

11. The side of a square $= a$; the origin of coördinates is the intersection of the diagonals. What are the coördinates of the vertices (i) if the axes are parallel to the sides of the square? (ii) if the axes coincide with the diagonals?

Ans. (i) $\left(\frac{a}{2}, \frac{a}{2}\right)$, $\left(-\frac{a}{2}, \frac{a}{2}\right)$, $\left(-\frac{a}{2}, -\frac{a}{2}\right)$, $\left(\frac{a}{2}, -\frac{a}{2}\right)$;

(ii) $\left(\frac{a}{2}\sqrt{2}, 0\right)$, $\left(0, \frac{a}{2}\sqrt{2}\right)$, $\left(-\frac{a}{2}\sqrt{2}, 0\right)$, $\left(0, -\frac{a}{2}\sqrt{2}\right)$.

12. The side of an equilateral triangle $= a$; the origin is taken at one vertex, and the axis of x coincides with one side. What are the coördinates of the three vertices?

Ans. $(0, 0)$, $(a, 0)$, $\left(\frac{a}{2}, \frac{a}{2}\sqrt{3}\right)$.

13. The line joining two points is bisected at the origin. If the coördinates of one of the points are a and b, what are the coördinates of the other?

14. Connect the points $(5, 3)$ and $(5, -3)$ by a straight line. What is the direction of this line?

Circular Measure.

5. In Analytic Geometry, angles are often expressed in degrees, minutes, and seconds; but sometimes it is very convenient to employ the *Circular Measure* of an angle.

In circular measure, an angle is defined by the equation

$$\text{angle} = \frac{\text{arc}}{\text{radius}},$$

in which the word "arc" denotes the length of the arc corresponding to the angle when both arc and radius are expressed in terms of a common linear unit.

This equation gives us a correct measure of angular magnitude, because (as shown in Geometry) for a given angle the value of the above ratio of arc and radius is constant for all values of the radius.

If the radius $= 1$, the equation becomes

$$\text{angle} = \text{arc}; \text{ that is,}$$

In circular measure an angle is measured by the length of the arc subtended by it in a unit circle.

It is shown in Geometry that the circumference of a unit circle $= 2\pi$; as this circumference contains 360° common measure, the two measures are easily compared by means of the relation

$$360 \text{ degrees} = 2\pi \text{ units, circular measure.}$$

Exercise 2.

1. Find the value in circular measure of the angles 1°, 45°, 90°, 180°, 270°.

Ans. $\frac{\pi}{180}, \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

2. In circular measure, the unit angle is that angle whose arc is equal to the radius of the circle. What is the value of this angle in degrees, etc.?

Ans. 57° 17′ 45″.

Distance between Two Points.

6. *To find the distance between two given points.*

Let P and Q (Fig. 2) be the given points, x_1 and y_1 the coördinates of P, x_2 and y_2 those of Q. Also let $d = PQ$ = the required distance.

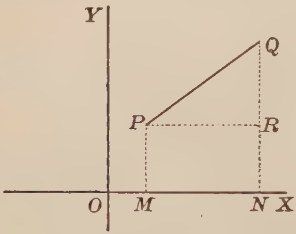


Fig. 2.

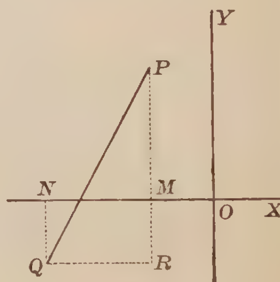


Fig. 3.

Draw PM and $QN \parallel$ to OY, and $PR \parallel$ to OX.

Then $\quad OM = x_1, \qquad MP = y_1,$

$\qquad ON = x_2, \qquad NQ = y_2,$

$\qquad PR = x_2 - x_1 \qquad QR = y_2 - y_1.$

By Geometry,

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2;$$

whence,
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \qquad [1]$$

Since $(x_1 - x_2)^2 = (x_2 - x_1)^2$, it makes no difference which point is called (x_1, y_1) and which (x_2, y_2).

7. Equation [1] is perfectly general, holding true for points situated in any quadrant. Thus, if P is in the second quadrant and Q in the third quadrant (Fig. 3), $x_2 - x_1$ is obviously equal to the leg RQ; and since y_2 is negative, $y_2 - y_1$ is the sum of two negative numbers, and is equal to the absolute length of the leg RP with the — sign prefixed.

NOTE. The learner should satisfy himself that equation [1] is perfectly general, by constructing other special cases in which the points P and Q are in different quadrants. In every case he will find that the numerical values of the expressions $(x_2 - x_1)$ and $(y_2 - y_1)$ are the legs of the right triangle, the hypotenuse of which is the required distance PQ.

Equation [1] is merely an illustration of the general truth that *theorems and formulas deduced by reasoning with points or lines in the first quadrant* (*where the coördinates are always positive*) *must, from the very nature of the analytic method, hold true when the points or lines are situated in the other quadrants.*

Exercise 3.

Find the distance

1. From the point $(-2, 5)$ to the point $(-8, -3)$.
2. From the point $(1, 3)$ to the point $(6, 15)$.
3. From the point $(-4, 5)$ to the point $(0, 2)$.
4. From the origin to the point $(-6, -8)$.
5. From the point (a, b) to the point $(-a, -b)$.

Find the lengths of the sides of a triangle

6. If the vertices are the points $(15, -4)$, $(-9, 3)$ $(11, 24)$.
7. If the vertices are the points $(2, 3)$, $(4, -5)$, $(-3, -6)$.
8. If the vertices are the points $(0, 0)$, $(3, 4)$, $(-3, 4)$.
9. If the vertices are the points $(0, 0)$, $(-a, 0)$, $(0, -b)$.
10. The vertices of a quadrilateral are $(5, 2)$, $(3, 7)$, $(-1, 4)$, $(-3, -2)$. Find the lengths of the sides and also of the diagonals.
11. One end of a line whose length is 13 is the point $(-4, 8)$; the ordinate of the other end is 3. What is its abscissa?
12. What equation must the coördinates of the point (x, y) satisfy if its distance from the point $(7, -2)$ is equal to 11?

13. What equation expresses algebraically the fact that the point (x, y) is equidistant from the points $(2, 3)$ and $(4, 5)$?

14. If the value of a quantity depends on the *square* of a length, it is immaterial whether the length is considered positive or negative. Why?

Division of a Line.

8. *To bisect the line joining two given points.*

Let P and Q (Fig. 4) be the given points (x_1, y_1) and (x_2, y_2). Let x and y be the coördinates of R, the mid-point of PQ.

The meaning of the problem is to find the values of x and y in terms of x_1, y_1, and x_2, y_2.

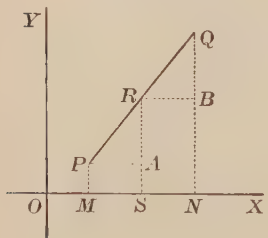


Fig. 4.

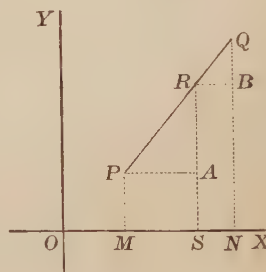


Fig. 5.

Draw PM, RS, $QN \parallel$ to OY; also draw PA, $RB \parallel$ to OX.

Then rt. $\triangle PRA =$ rt. $\triangle RQB$ (hypotenuse and one acute angle equal).

Therefore, $PA = RB$, and $AR = BQ$;

also, $MS = SN$.

By substitution, $x - x_1 = x_2 - x$, and $y - y_1 = y_2 - y$;

whence,
$$x = \frac{x_1 + x_2}{2};\ y = \frac{y_1 + y_2}{2}. \qquad [2]$$

9. *To divide the line joining two given points into two parts having a given ratio m : n.*

Let P and Q (Fig. 5) be the given points $(x_1\ y_1)$ and $(x_2\ y_2)$. Let R be the required point, such that $PR : RQ = m : n$, and let x and y denote the coördinates of R.

Complete the figure by drawing lines as in Fig. 4.

The rt. $\triangle$ PRA and RQB, being mutually equiangular, are similar; therefore

$$\frac{PA}{RB} = \frac{PR}{RQ} = \frac{m}{n}, \text{ and } \frac{AR}{BQ} = \frac{PR}{RQ} = \frac{m}{n}.$$

Substituting for the lines their values, we have

$$\frac{x - x_1}{x_2 - x} = \frac{m}{n}, \text{ and } \frac{y - y_1}{y_2 - y} = \frac{m}{n}.$$

Solving these equations for x and y, we obtain

$$x = \frac{mx_2 + nx_1}{m + n};\ y = \frac{my_2 + ny_1}{m + n}. \qquad [3]$$

If $m = n$, we have the special case of bisecting a line already considered; and it is easy to see that the values of x and y reduce to the forms given in [2].

Exercise 4.

What are the coördinates of the point

1. Halfway between the points (5, 3) and (7, 9)?
2. Halfway between the points (−6, 2) and (4, −2)?
3. Halfway between the points (5, 0) and (−1, −4)?
4. The vertices of a triangle are (2, 3), (4, −5), (−3, −6); find the middle points of its sides.
5. The middle point of a line is (6, 4), and one end of the line is (5, 7). What are the coördinates of the other end?
6. A line is bisected at the origin; one end of the line is the point $(-a, b)$. What are the coördinates at the other end?

7. Prove that the middle point of the hypotenuse of a right triangle is equidistant from the three vertices.

8. Prove that the diagonals of a parallelogram mutually bisect each other.

9. Show that the values of x and y in [2] hold true when the two given points both lie in the second quadrant.

10. Solve the problem of § 9 when the line PQ is cut *externally* instead of internally, in the ratio $m : n$.

11. What are the coördinates of the point that divides the line joining $(3, -1)$ and $(10, 6)$ in the ratio $3 : 4$?

12. The line joining $(2, 3)$ and $(4, -5)$ is trisected. Determine the point of trisection nearer $(2, 3)$.

13. A line AB is produced to a point C, such that $BC = \frac{1}{2} AB$. If A and B are the points $(5, 6)$ and $(7, 2)$, what are the coördinates of C?

14. A line AB is produced to a point C, such that $AB : BC = 4 : 7$. If A and B are the points $(5, 4)$ and $(6, -9)$, what are the coördinates of C?

15. Three vertices of a parallelogram are $(1, 2)$, $(-5, -3)$, $(7, -6)$. What is the fourth vertex?

Constants and Variables.

10. In Analytic Geometry a line is regarded as *a geometric magnitude traced or generated by a moving point*, — just as we trace on paper what serves to represent a line to the eye by moving the point of a pen or pencil over the paper.

We shall find that great advantages are to be gained by defining a line in this way, but we must be prepared from the outset to make an important distinction in the use of symbols representing lengths. We must distinguish between symbols which denote definite or fixed lengths and those which denote *variable* lengths.

11. Let A (Fig. 6) be the point (3, 4). Then $OA = \sqrt{9+16} = 5$. Now let a point P describe the line OA by moving from O to A, and let the coördinates of P be denoted by x and y; also let z denote the length OP at any position of P. Then z will increase continuously from 0 to 5.

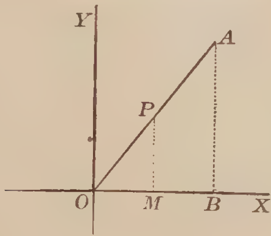


Fig. 6.

Here the word *continuously* deserves special attention. It means that P must pass successively through every position on the line OA from O to A; that, therefore, z must have in succession every conceivable value between 0 and 5. There will be one position of P for which z is equal to 2; there will be another position of P for which z is equal to 2.000001; but before reaching this value z must first pass through all values between 2 and 2.000001.

In the same way x and y, the coördinates of P, both pass through a continuous change, x increasing continuously from 0 to 3, and y from 0 to 4.

We may now divide the lengths considered in this example into two classes:

(1) Lengths supposed to remain constant in value, namely, the coördinates of A and the distance OA; (2) lengths supposed to vary continuously in value, namely, the coördinates of P (x and y), and the distance OP, or z.

Quantities of the first kind in any problem are called *constant* quantities, or, more briefly, **Constants.**

Quantities of the second kind are called variable quantities, or, more briefly, **Variables.**

12. Two variables are often so related that if one of them changes in value the other also changes in value. The second variable is then said to be a *function* of the first variable. The second variable is also called the *dependent* variable, while the first is called the *independent* variable. Usually the relation between two variables is such that either may be treated as the independent variable, and the other as the dependent variable.

Thus, in § 11, if we suppose z to change, then both x and y will change; the values of x and y then will depend upon the value given to z; that is, x and y will be *functions* of z. But we may also suppose the value of x, the abscissa of P, to change; then it is clear that the values of both y and z must also change. In this case we take x as the independent variable, and values of y and z will depend upon the value of x; that is, y and z will be *functions* of x.

13. The most concise way to express the relations of the constants and variables which enter into a problem is by means of *algebraic equations.*

The coördinates of P (Fig. 6) throughout its motion are always x and y; and the triangle OPM is similar to the triangle OAB. Hence, for any position of P,

$$\frac{y}{x}=\frac{4}{3}, \text{ and } z^2=x^2+y^2.$$

By solving, $$y=\frac{4}{3}x, \text{ and } z=\frac{5}{3}x,$$

These equations express the values of y and z, respectively, in terms of x as the independent variable.

14. In § 11, instead of assuming 3 and 4 as the coördinates of A, we might have employed two letters, as a and b, with the understanding that these letters should denote two coördinates that remain *constant in value* during the motion of P. If we choose these letters, we obtain,

$$y = \frac{b}{a}x, \quad z = \frac{\sqrt{a^2 + b^2}}{a}x.$$

15. There is a noteworthy difference between the constants 3 and 4 and the constants a and b. The numbers 3 and 4 cannot be supposed to change under any circumstances. The numbers a and b are constants in this sense only, that they do not change in value when we suppose x or y or z to change in value; in other words, they are not functions of x or y or z in the particular problem under discussion. In all other respects they are free to represent as many different values as we choose to assign to them.

Constants of the first kind (arithmetical numbers) are called *absolute* constants. Constants of the second kind (letters) are called *arbitrary* or *general* constants.

16. By general agreement, variables are represented by the last letters of the alphabet, as x, y, z; while constants are represented by the first letters, a, b, c; or by the last letters with subscripts, as x_1, y_1, x_2, y_2, etc.

Exercise 5.

1. A point P (x, y) revolves about the point Q (x_1, y_1), keeping always at the distance a from it. Name the constants and the variables in this case. What is the total change in the value of each variable?

2. A point Q (x, y) moves: first parallel to the axis of y, then parallel to the axis of x, then equally inclined to the axes. Point out in each case the constants and the variables.

Locus of an Equation.

17. Let us continue to regard x and y as the coördinates of a point, and proceed to illustrate the meaning of an algebraic equation containing one or both of these letters.

Take as the first case the equation $x-4=0$, whence $x=4$. It is clear that this equation is satisfied by the coördinates of every point so situated that its abscissa is equal to 4; therefore, it is satisfied by the coördinate of

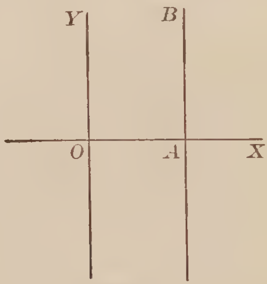


Fig. 7.

every point in the line AB (Fig. 7), drawn ∥ to OY, on the right of OY, and at the distance 4 from OY. And it is also clear that this line contains *all* the points whose coördinates will satisfy the given equation.

The line AB, then, may be regarded as the *geometric* representation or meaning of the equation $x-4=0$; and, conversely, the equation $x-4=0$ may be considered to be the *algebraic* representative of this particular line.

In Analytic Geometry the line AB is called the **locus** of the equation $x-4=0$; conversely, the equation $x-4=0$ is known as the **equation** of the line AB.

The line AB is to be regarded as extending indefinitely in both directions. If AB is described by a point P, moving parallel to the axis of y, then at all points x is

constant in value and equal to 4, while y (which does not appear in the given equation) is a variable, passing through an unlimited number of values, both positive and negative.

18. The equation $x-y=0$, or $x=y$, states in algebraic language that the abscissa of the point is always equal to the ordinate.

Values of x.	Values of y.
0	0.
1	1.
2	2.
-1	-1.
etc.	etc.

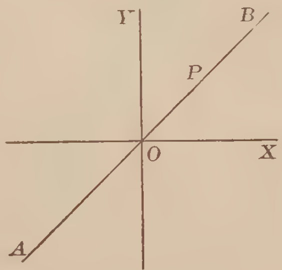


Fig. 8.

If we draw through the origin O (Fig. 8) a straight line AB, bisecting the first and third quadrants, then it is easy to see that the given equation is satisfied *by every point* in this line and *by no other* points. If we conceive a point P to move so that its abscissa shall always be equal to its ordinate, then the point must describe the line AB. In other words, if the point P is obliged to move so that its coördinates (which of course are variables) shall always satisfy the condition expressed by the equation $x-y=0$; then the motion of P is confined to the line AB.

The line AB is the locus of the equation $x-y=0$, and this equation represents the line AB.

19. The equation $2x+y-3=0$ is satisfied by an unlimited number of values of x and y. We may find as many of them as we please by assuming values for one of the variables, and computing the corresponding values of the other.

If we assume for x the values given below, we easily find for y the corresponding values given in the next column.

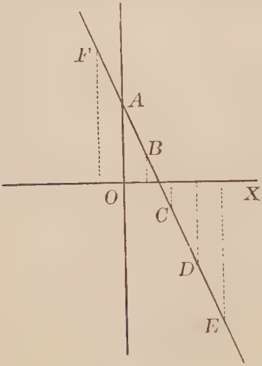


Fig. 9.

Values of x.	Values of y.
0	3.
1	1.
2	-1.
3	-3.
4	-5.
-1	5.
-2	7.
-3	9.
-4	11.
etc.	etc.

Plotting these points (as shown in Fig. 9), we obtain a series of points so placed that their coördinates all satisfy the given equation. By assuming for x values between 0 and 1, 1 and 2, etc., we might in the same way obtain as many points as we please between A and B, B and C, etc. In this case, however, the points all lie in a *straight* line (as will be shown later); so that if any *two* points are found, the straight line drawn through them will include *all* the points whose coördinates satisfy the given equation. Now imagine that a point P, the coördinates of which are denoted by x and y, is required to move in such a way that the values of x and y shall always satisfy the equation $2x + y - 3 = 0$; then P *must* describe the line AB, and cannot describe any other line.

The line AB is the locus of the equation $2x + y - 3 = 0$.

20. Thus far we have taken equations of the first degree. Let us now consider the equation $x^2 - y^2 = 0$. By solving for y, we obtain $y = \pm x$. Hence, for every

value of x there are *two* values of y, both equal numerically to x, but having unlike signs. Thus, for assumed values of x, we have corresponding valués of y given below:

Values of x.	Values of y.
0	0.
1	1, −1.
2	2, −2.
3	3, −3.
−1	−1, 1.
−2	−2, 2.
−3	−3, 3.

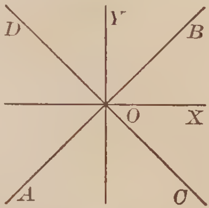


Fig. 10.

By plotting a few points, and comparing this case with the example in § 18, it becomes evident that the locus of the equation consists of *two* lines, AB, CD (Fig. 10), drawn through the origin so as to bisect the four quadrants.

21. There is another way of looking at this case. The equation $x^2-y^2=0$, by factoring, may be written $(x-y)(x+y)=0$. Now the equation is satisfied if *either* factor $=0$; hence, it is satisfied if $x-y=0$, and also if $x+y=0$. We know (see § 18) that the locus of the equation $x-y=0$ is the line AB (Fig. 8). And the locus of the equation $x+y=0$ (or $x=-y$) is evidently the line CD, since every point in it is so placed that the two coördinates are equal numerically but unlike in sign. Therefore, the original equation $x^2-y^2=0$ is represented by the pair of lines AB and CD (Fig. 10).

22. Let us next consider the equation $x^2+y^2=25$. Solving for y, we obtain $y=\pm\sqrt{25-x^2}$. When $x<5$ there are two values of y equal numerically but unlike in sign. When $x=5$, $y=0$. When $x>5$ the values of y are imaginary; this last result means that there is no point with an abscissa greater than 5 whose coördinates will satisfy the given equation.

By assigning values of x differing by unity, we obtain the following sets of values of x and y; and by plotting the points, and then drawing through them a continuous curve, we obtain the curve shown in Fig. 11.

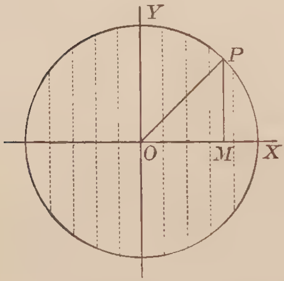


Fig. 11.

Values of x.	Values of y.
0	$\pm 5.$
1	$\pm 4.9.$
2	$\pm 4.6.$
3	$\pm 4.$
4	$\pm 3.$
5	0.
-1	$\pm 4.9.$
-2	$\pm 4.6.$
-3	$\pm 4.$
-4	$\pm 3.$
-5	0.

In this case, however, the locus may be found as follows:

Let P (Fig. 11) be any point so placed that its coördinates, $x = OM$, $y = MP$, satisfy the equation $x^2 + y^2 = 25$. Join OP; then $x^2 + y^2 = \overline{OP}^2$; therefore, $OP = 5$. Hence, if P is any point in the *circumference* described with O as centre and 5 for radius, its coördinates will satisfy the given equation; and if P is *not* in this circumference, its coördinates will *not* satisfy the equation. This circumference, then, is the locus of the equation.

23. The points whose coördinates satisfy the equation $y^2 = 4x$ lie neither in a straight line nor in a circumference. Nevertheless, they do all lie in a certain line, which is, therefore, completely determined by the equation. To construct this line, we first find a number of points that satisfy the equation (the closer the points to one another, the better) and then draw, freehand or with the aid of tracing curves, a continuous curve through the points.

The coördinates of a number of such points are given in the table below. It is evident that for each positive value of x there are two values of y, equal numerically but unlike in sign. For a negative value of x, the value of y is imaginary; this means that there are no points to the left of the axis of y that will satisfy the given equation.

Values of x.	Values of y.
0	0.
1	± 2.
2	± 2.83.
3	± 3.46.
4	± 4.
5	± 4.47.
6	± 4.90.
7	± 5.29.
8	± 5.66.
9	± 6.
-1	imaginary.

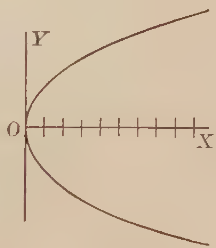


Fig. 12.

In Fig. 12 the several points obtained are plotted, and a smooth curve is then drawn through them. It passes through the origin, is placed symmetrically on both sides of the axis of x, lies wholly on the right of the axis of y, and extends towards the right without limit. It is the locus of the given equation, and is a curve called the **Parabola.**

24. After a study of the foregoing examples, we may lay down the following general principles, which form the foundation of the science of Analytic Geometry:

I. Every algebraic equation involving x and y is satisfied by an unlimited number of sets of values of x and y; in other words, x and y may be treated as *variables*, or quantities varying continuously, yet always so related that their values constantly satisfy the equation.

II. The letters x and y may also be regarded as representing the coördinates of a point. This point is not fixed in position, because x and y are variables; but it cannot be placed at random, because x and y can have only such values as will satisfy the equation; now, since these values are continuous, the point may be conceived to *move continuously*, and will therefore describe a definite line, or group of lines.

The line, or group of lines, described by a point moving so that its coördinates always satisfy the equation is called the **Locus of the Equation**; conversely, the equation satisfied by the coördinates of every point in a certain line is called the **Equation of the Line.**

An *equation*, therefore, containing the variables x and y is the algebraic representation of a line.

In *Analytic Geometry* the loci considered are represented by their equations, and the investigation of their properties is carried on by means of these equations.

Exercise 6.

Determine and construct the loci of the following equations (the locus in each case being either a straight line or a circumference of a circle):

1. $x - 6 = 0$.
2. $x + 5 = 0$.
3. $y = -7$.
4. $x = 0$.
5. $y = 0$.
6. $x + y = 0$.
7. $x - 2y = 0$.
8. $2x + 3y + 10 = 0$.
9. $9x^2 - 25 = 0$.
10. $4x^2 - y^2 = 0$.
11. $x^2 - 16y^2 = 0$.
12. $x^2 + y^2 = 36$.
13. $x^2 + y^2 - 1 = 0$.
14. $x(y + 5) = 0$.
15. $(x - 2)(x - 3) = 0$.
16. $(y - 4)(y + 1) = 0$.

17. What is the geometric meaning of the equation $5x^2-17x-12=0$?

HINT. Resolve the equation into two binomial factors.

18. What is the geometric meaning of the equation $y^2+3y=0$?

19. What two lines form the locus of the equation $xy+4x=0$?

20. Is the point $(2, -5)$ situated in the locus of the equation $4x-3y-22=0$?

HINT. See if the coördinates of the point satisfy the equation.

21. Is the point $(4, -6)$ in the locus of the equation $y^2=9x$?

22. Is the point $(-1, -1)$ in the locus of the equation $16x^2+9y^2+15x-6y-18=0$?

23. Does the locus of the equation $x^2+y^2=100$ pass through the point $(-6, 8)$?

24. Which of the loci represented by the following equations pass through the origin?

(1) $3x+2=0$. (5) $3x=2y$.

(2) $3x-11y+7=0$. (6) $3x-11y=0$.

(3) $x^2-16y^2-10=0$. (7) $x^2-16y^2=0$.

(4) $ax+by+c=0$. (8) $ax+by=0$.

25. The abscissa of a point in the locus of the equation $3x-4y-7=0$ is 9; what is the value of the ordinate?

Ans. 5.

26. Determine that point in the locus of $y^2-4x=0$ for which the ordinate $=-6$.

Ans. The point $(9, -6)$.

27. Determine the point where the line represented by the equation $7x+y-14=0$ cuts the axis of x.

Ans. The point $(2, 0)$.

Intersections of Loci.

25. The term **Curve,** as used in Analytic Geometry, means any geometric locus, including the straight line as well as lines commonly called curves.

The **Intercepts** of a curve on the axes are the distances from the origin to the points where the curve cuts the axes.

26. *To find the intercepts of a curve, having given its equation.*

The intercept of a curve on the axis of x is the abscissa of the point where the curve cuts the axis of x. The ordinate of this point $= 0$. Therefore, to find this intercept, put $y = 0$ in the given equation of the curve, and then solve the equation for x; the resulting real values of x will be the intercepts required.

If the equation is of a higher degree than the first, there will in general be more than one real value of x; and the curve will intersect the axis of x in as many points as there are *real* values of x.

To an imaginary value of x there corresponds no intercept; but it is sometimes convenient to speak of such a value as an imaginary intercept.

Similarly, to find the intercepts on the axis of y, put $x = 0$ in the given equation, and then solve it for y; the resulting real values of y will be the intercepts required.

27. *To find the points of intersection of two curves, having given their equations.*

Since the points of intersection lie in both curves, their coördinates must satisfy both equations. Therefore, to find their coördinates, solve the two equations, regarding the variables x and y as unknown quantities.

If the equations are both of the first degree, there will

be only *one* pair of values of x and y, and one point of intersection.

If the equations are, one or both of them, of higher degree than the first, there may be several pairs of values of x and y; in this case there will be as many points of intersection as there are pairs of *real* values of x and y.

If imaginary values of either x or y are obtained, there are no corresponding points of intersection.

28. *If a curve passes through the origin, its equation, reduced to its simplest form, cannot have a constant term; that is, cannot have a term free from both x and y.*

Since in this case the point (0, 0) is a point of the curve, its equation must be satisfied by the values $x=0$, and $y=0$. But it is obvious that these values cannot satisfy the equation if, after reduction to its simplest form, it still contains a constant term. Therefore the equation cannot have a constant term.

29. *If an equation has no constant term, its locus must pass through the origin.*

For, the values $x=0$, $y=0$ must evidently satisfy the equation, and therefore the point (0, 0) must be a point of the locus.

Exercise 7.

Find the intercepts of the following curves:

1. $4x+3y-48=0$.
2. $5y-3x-30=0$.
3. $x^2+y^2=16$.
4. $9x^2+4y^2=16$.
5. $9x^2-4y^2=16$.
6. $9x^2-4y=16$.
7. $a^2x^2+b^2y^2=a^2b^2$.
8. $x-3=0$.
9. $x^2-9=0$.
10. $x^2-y^2=0$.
11. $y^2=4x$.
12. $x^2+y^2-4x-8y=32$.
13. $x^2+y^2-4x-8y=0$.
14. $(x-5)^2+(y-6)^2=20$.

Find the points of intersection of the following curver:

15. $3x-4y+13=0$, $11x+7y-104=0$.

16. $2x+3y=7$, $x-y=1$.

17. $x-7y+25=0$, $x^2+y^2=25$.

18. $3x+4y=25$, $x^2+y^2=25$.

19. $x+y=8$, $x^2+y^2=34$.

20. $2x=y$, $x^2+y^2-10x=0$.

21. The equations of the sides of a triangle are $2x+9y+17=0$, $7x-y-38=0$, $x-2y+2=0$. Find the coördinates of its three vertices.

22. The equations of the sides of a triangle are $5x+6y=12$, $3x-4y=30$, $x+5y=10$. Find the lengths of its sides.

23. Find the lengths of the sides of a triangle if the equations of the sides are $x=0$, $y=0$, and $4x+3y=12$.

24. What are the vertices of the quadrilateral enclosed by the straight lines $x-a=0$, $x+a=0$, $y-b=0$, $y+b=0$? What kind of a quadrilateral is it?

25. Does the straight line $5x+4y=20$ cut the circle $x^2+y^2=9$?

26. Find the length of that part of the straight line $3x-4y=0$ which is contained within the circle $x^2+y^2=25$.

27. Which of the following curves pass through the origin of coördinates?

(1) $7x-2y+4=0$.	(4) $ax+by=0$.
(2) $7x-2y=0$.	(5) $ax+by+c=0$.
(3) $y^2-x^2=4y$.	(6) $x^2-y+a=a+xy$.

28. Change the equation $4x+2y-7=0$ so that its locus shall pass through the origin.

Construction of Loci.

30. If we know that the locus of a given equation is a straight line, the locus is easily constructed; it is only necessary to find any two points in it, plot them, and draw a straight line through them with the aid of a ruler.

Likewise, if we know that the locus is a circumference, and can find its centre and its radius, the entire locus can then be described with the aid of a pair of compasses.

It will appear later that the *form* of the given equation enables us at once to tell whether its locus is a straight line or a circumference.

If the locus of an equation is neither a straight line nor a circumference, then the following method of construction, which is applicable to the locus of any equation without regard to the form of the curve, is usually employed.

31. *To construct the locus of a given equation.*

The steps of the process are as follows:

1. Solve the equation with respect to either x or y.

2. Assign values to the other variable, differing not much from one another.

3. Find each corresponding value of the first variable.

4. Draw two axes, choose a suitable scale of lengths, and plot the points whose coördinates have been obtained.

5. Draw a continuous curve through these points.

Discussion. An examination of the equation, as shown in the examples given below, enables us to obtain a good general idea of the shape and size of the curve, its position with respect to the axes, etc.; in this way it serves as an aid in constructing the curve, and as a means of detecting numerical errors made in computing the coördinates of the points. Such an examination is called a **discussion** of the equation.

NOTE 1. This method of constructing a locus is from its nature an *approximate* method. But the nearer the points are to one another, the nearer the curve will approach the exact position of the locus.

NOTE 2. In theory, it is immaterial what scale of lengths is used. In practice, the unit of lengths should be determined by the size of the paper compared with the greatest length to be laid off upon it. Paper sold under the name of "*coördinate* paper," ruled in small squares, $\frac{1}{10}$ of an inch on a side, will be found very convenient in practice.

32. Construct the locus of the equation

$$9x^2 + 4y^2 - 576 = 0.$$

If we solve for both x and y, we obtain the following values:

$$y = \pm \tfrac{3}{2}\sqrt{64 - x^2}, \qquad (1)$$
$$x = \pm \tfrac{2}{3}\sqrt{144 - y^2}. \qquad (2)$$

By assigning to x values differing by unity, and finding corresponding values of y, we obtain the results given below. To each value of x, positive or negative, there correspond two values of y, equal numerically and unlike in sign. By plotting the corresponding points, and drawing a continuous curve through them, we obtain the closed curve shown in Fig. 13.

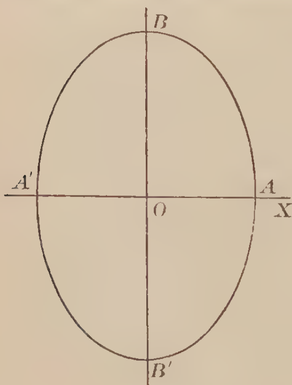


Fig. 13.

Values of x.	Values of y.
0	$\pm 12.$
± 1	$\pm 11.91.$
± 2	$\pm 11.62.$
± 3	$\pm 11.13.$
± 4	$\pm 10.39.$
± 5	$\pm 9.36.$
± 6	$\pm 7.93.$
± 7	$\pm 5.80.$
± 8	$\pm 0.$
± 9	$\pm$ imaginary.

DISCUSSION. From equations (1) and (2) we see that if $x=0$, $y=\pm 12$, and if $y=0$, $x=\pm 8$; therefore, the intercepts of the curve on the axis of x are $+8$ and -8, and those on the axis of y are $+12$ and -12. These intercepts are the lengths OA, OA', and OB, OB', in Fig. 13.

If we assign to x a numerical value greater than 8, positive or negative, we find by substitution in equation (1) that the corresponding value of y will be imaginary. This shows that OA and OA' are the maximum abscissas of the curve. Similarly, equation (2) shows that the curve has no points with ordinates greater than $+12$ and -12.

The greater the numerical value of x, between the limits 0 and $+8$ or 0 and -8, the less the corresponding value of y numerically; why?

From equation (1) we see that for each value of x, between the limits 0 and ± 8, there are *two* real values of y, equal numerically and unlike in sign. Hence, for each value of x between 0 and ± 8 there are *two* points of the curve placed equally distant from the axis of x. Therefore, the curve is *symmetrical* with respect to the axis of x; in other words, if the portion of the curve above the axis of x is revolved about this axis through 180°, it will coincide with the portion below the axis. Similarly, it follows from equation (2) that the curve is also symmetrical with respect to the axis of y. Therefore, the entire curve is a closed curve, consisting of four equal quadrantal arcs symmetrically placed about the origin O. The name of this curve is the **Ellipse.**

33. Construct the locus of the equation

$$4x-y^2+16=0.$$

Solving for both x and y, we obtain

$$y=\pm 2\sqrt{x+4}, \tag{1}$$

$$x=\frac{y^2-16}{4}. \tag{2}$$

We may either assign values to x, and then compute values of y by means of (1), or assign values to y, and compute values of x by means of (2); the second course is better, because there is less labor in squaring a number than in extracting its square root.

By assigning values to y, differing by unity from 0 to $+10$, and from 0 to -10, and then proceeding exactly as in the last example, we obtain the series of values given below, and the curve shown in Fig 14.

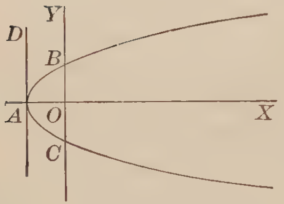


Fig. 14.

Values of y.	Values of x.
± 0	$-4.$
± 1	$-3.75.$
± 2	$-3.$
± 3	$-1.75.$
± 4	$0.$
± 5	$2.25.$
± 6	$5.$
± 7	$8.25.$
± 8	$12.$
± 9	$16.25.$
± 10	$21.$

Discussion. An examination of equations (1) and (2) yields the following results, the reasons for which are left as an exercise for the learner:

The intercepts on the axes are:

On the axis of x, $OA = -4$:

On the axis of y, $OB = +4$, and $OC = -4$.

If we draw through A the line $AD \perp$ to OX, the entire curve lies to the right of AD.

The curve is situated on both sides of OX, and is symmetrical with respect to OX.

The curve extends towards the right without limit.

The curve constantly recedes from OX as it extends towards the right.

This curve is called a **Parabola**; the point A is called its **Vertex**; the line AX its **Axis**.

34. Construct the locus of the equation

$$y = \sin x.$$

If we assume for x the values 0°, 10°, 20°, 30°, etc., the corresponding values of y are the *natural* sines of these angles, and are as follows:

Values of x.	Values of y.	Values of x.	Values of y.
0°	0.	50°	0.77.
10°	0.17.	60°	0.87
20°	0.34.	70°	0.94.
30°	0.50.	80°	0.98.
40°	0.64.	90°	1.

If we continue the values of x from 90° to 180°, the above values of y repeat themselves in the inverse order (*e.g.*, if $x = 100°$, $y = 0.98$, etc.); from 180° to 360° the values of y are numerically the same, and occur in the same order as between 0° and 180°, but are negative.

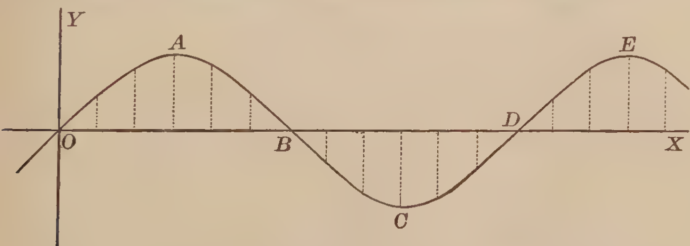


Fig. 15.

In order to express both x and y in terms of a common linear unit, we ought, in strictness, to use the circular measure of an angle in which the linear unit represents an angle

of 57.3°, very nearly (see § 5). But it is more convenient, and serves our present purpose equally well, to assume that an angle of 60° = the linear unit. This assumption is made in Fig. 15, where the curve is drawn with one centimeter as the linear unit.

Discussion. The curve passes through the origin, and cuts the axis of x at points separated by intervals of 180°. Since an angle may have any magnitude, positive or negative, the curve extends on both sides of the origin without limit. The maximum value of the ordinate is alternately $+1$ and -1: the former value corresponds to the angle 90°, and repeats itself at intervals of 360°; the latter value corresponds to the angle 270°, and repeats itself at intervals of 360°. The curve has the form of a wave, and is called the **Sinusoid.**

Exercise 8.

Construct the loci of the following equations:

1. $3x - y - 2 = 0.$
2. $y = 2x.$
3. $x^2 = y^2.$
4. $x^2 + y^2 = 100.$
5. $x^2 - y^2 = 25.$
6. $4x^2 - y^2 = 0.$
7. $4x^2 + 9y^2 = 144.$
8. $y^2 - 16x = 0.$
9. $y^2 + 16x = 0.$
10. $x^2 - 2x - 10y - 5 = 0.$
11. $y^2 - 2y - 10x = 0.$
12. $(x-3)^2 + (y-2)^2 = 25.$
13. $y^2 - 1 = 0.$
14. $y = x^3.$
15. $xy = 12.$
16. $x = \sin y.$
17. $y = 2 \sin x.$
18. $y = \sin 2x.$
19. $y = \cos x.$
20. $y = \tan x.$
21. $y = \cot x.$
22. $y = \sec x.$
23. $y = \csc x.$
24. $y = \sin x + \cos x.$

Equation of a Curve.

35. From what precedes, we may conclude that every equation involving x and y as variables represents a definite line (or group of lines) known as the *locus* of the equation. Regarded from this point of view, an equation is the statement in algebraic language of a *geometric condition* which must always be satisfied by a point (x, y), as we imagine it to move in the plane of the axes. For example, the equation $x = 2y$ states the condition that the point must so move that its abscissa shall always be equal to twice its ordinate; the equation $x^2 + y^2 = 4$ states the condition that the point must so move that the sum of the squares of its coördinates shall always be equal to 4; etc.

Conversely, every geometric condition that a point is required to satisfy must confine the point to a definite line as its locus, and must lead to an equation that is always satisfied by the coördinates of the point.

Hence arises a new problem, and one usually of greater difficulty than any thus far considered, namely:

Given the geometric condition to be satisfied by a point, to find the equation of its locus.

The importance of this problem is that in the practical applications of Analytic Geometry the law of a moving point is commonly the one thing known, so that the first step must consist in finding the equation of its locus.

Exercise 9.

1. A point moves so that it is always three times as far from the axis of x as from the axis of y. What is the equation of its locus?

2. What is the equation of the locus of a point that moves so that its abscissa is always equal to $+6$? -6? 0?

3. What is the equation of the locus of a point that moves so that its ordinate is always equal to $+4$? -1? 0?

4. A point so moves that its distance from the straight line $x=3$ is always numerically equal to 2. What is the equation of its locus?

5. A point so moves that its distance from the straight line $y=5$ is always numerically equal to 3. Find the equation of its locus. Construct the locus.

6. A point moves so that its distance from the straight line $x+4=0$ is always numerically equal to 5. Find the equation of its locus. Construct the locus.

7. What is the equation of the locus of a point equidistant

(1) from the parallels $x=0$ and $x=-6$?
(2) from the parallels $y=7$ and $y=-3$?

8. What is the equation of the locus of a point always equidistant from the origin and the point $(6, 0)$?

Find the equation of the locus of a point

9. Equidistant from the points $(4, 0)$ and $(-2, 0)$.

10. Equidistant from the points $(0, -5)$ and $(0, 9)$.

11. Equidistant from the points $(3, 4)$ and $(5, -2)$.

12. Equidistant from the points $(5, 0)$ and $(0, 5)$.

13. A point moves so that its distance from the origin is always equal to 10. Find the equation of its locus.

14. A point moves so that its distance from the point $(4, -3)$ is always equal to 5. Find the equation of its locus, and construct it. What kind of curve is it? Does it pass through the origin? Why?

15. What is the equation of the locus of a point whose distance from the point $(-4, -7)$ is always equal to 8?

16. About the origin of coördinates as centre, with a radius equal to 5, a circle is described. A point outside this circle so moves that its distance from the circumference of the circle is always equal to 4. What is the equation of its locus?

17. A high rock A, rising out of the water, is 3 miles from a perfectly straight shore BC. A vessel so moves that its distance from the rock is always the same as its distance from the shore. What is the equation of its locus?

18. A point A is situated at the distance 6 from the line BC. A moving point P is always equidistant from A and BC. Find the equation of its locus.

19. A point moves so that its distance from the axis of x is half its distance from the origin; find the equation of its locus.

20. A point moves so that the sum of the squares of its distances from the two fixed points $(a, 0)$ and $(-a, 0)$ is the constant $2k^2$; find the equation of its locus.

21. A point moves so that the difference of the squares of its distances from $(a, 0)$ and $(-a, 0)$ is the constant k^2; find the equation of its locus.

Exercise 10. (Review.)

1. If we plot all possible points for which $x = -5$, how will they be situated?

2. Construct the point (x, y) if $x = 2$ and

$$(1)\ y = 4x - 3, \quad (2)\ 3x - 2y = 8.$$

3. The vertices of a rectangle are the points (a, b), $(-a, b)$, $(-a, -b)$, and $(a, -b)$. Find the lengths of its sides, the lengths of its diagonals, and show that the vertices are equidistant from the origin.

4. What does equation [1], p. 6, for the distance between two points, become when one of the points is the origin?

5. Express by an equation that the distance of the point (x, y) from the point $(4, 6)$ is equal to 8.

6. Express by an equation that the point (x, y) is equidistant from the points $(2, 3)$ and $(4, 5)$.

7. Find the point equidistant from the points $(2, 3)$, $(4, 5)$, and $(6, 1)$. What is the common distance?

8. Prove that the diagonals of a rectangle are equal.

9. Prove that the diagonals of a parallelogram mutually bisect each other.

10. The coördinates of three vertices of a parallelogram are known: $(5, 3)$, $(7, 10)$, $(13, 9)$. What are the coördinates of the remaining vertex?

11. The coördinates of the vertices of a triangle are $(3, 5)$, $(7, -9)$, $(2, -4)$. Find the coördinates of the middle points of its sides.

12. The centre of gravity of a triangle is situated on the line joining any vertex to the middle point of the opposite side, at the point of trisection nearer that side. Find the centre of gravity of the triangle whose vertices are the points $(2, 3)$, $(4, -5)$, $(-3, -6)$.

13. The vertices of a triangle are $(5, -3)$, $(7, 9)$, $(-9, 6)$. Find the distance from its centre of gravity to the origin.

14. If the vertices of a quadrilateral are $(0, 0)$, $(5, 0)$, $(9, 11)$, $(0, 3)$, what are the coördinates of the intersection of the two straight lines that join the middle points of the opposite sides?

15. Prove that the two straight lines which join the middle points of the opposite sides of any quadrilateral mutually bisect each other.

16. A line is divided into three equal parts. One end of the line is the point $(3, 8)$; the adjacent point of division is $(4, 13)$. What are the coördinates of the other end?

17. The line joining the points (x_1, y_1) and (x_2, y_2) is divided into four equal parts. Find the coördinates of the points of division.

18. Explain and illustrate the relation that exists between an equation and its locus.

19. Construct the two lines that form the locus of the equation $x^2 - 7x = 0$.

20. Is the point $(2, -5)$ in the locus of the equation $4x^2 - 9y^2 = 36$?

21. The ordinate of a certain point in the locus of the equation $x^2 + y^2 + 20x - 70 = 0$ is 1. What is the abscissa of this point?

22. Find the intercepts of the curve

$$x^2 + y^2 - 5x - 7y + 6 = 0.$$

Find the points common to the curves:

23. $x^2 + y^2 = 100$, and $y^2 - \frac{9x}{2} = 0$.

24. $x^2 + y^2 = 5a^2$, and $x^2 = 4ay$.

25. $b^2x^2 + a^2y^2 = a^2b^2$, and $x^2 + y^2 = a^2$.

26. Find the lengths of the sides of a triangle, if its vertices are $(6, 0)$, $(0, -8)$, $(-4, -2)$.

27. A point moves so that it is always six times as far from one of two fixed perpendicular lines as from the other. Find the equation of its locus.

28. A point so moves that its distance from the fixed point A is always double its distance from the fixed line AB. Find the equation of its locus.

29. A fixed point is at the distance a from a fixed straight line. A point so moves that its distance from the fixed point is always twice its distance from the fixed line. Find the equation of its locus.

CHAPTER II.

THE STRAIGHT LINE.

Equations of the Straight Line.

36. *Notation.* Throughout this chapter, and generally in equations of straight lines,

$a =$ the intercept on the axis of x.
$b =$ the intercept on the axis of y.
$\gamma =$ the angle between the axis of x and the line.
$m = \tan \gamma$.
$p =$ the perpendicular from the origin to the line.
$\alpha =$ the angle between the axis of x and p.

These six quantities are *general constants;* a, b, and m may have any values from $-\infty$ to $+\infty$; p, any value from 0 to $+\infty$; γ, any value from $0°$ to $180°$; α, any value from $0°$ to $360°$.

The constant m is often called the **Slope** of the line; its value determines the direction of the line.

In order to determine a straight line, *two* geometric conditions must be given.

37. *To find the equation of a straight line passing through two given points (x_1, y_1) and (x_2, y_2).*

Let A (Fig. 16) be the point (x_1, y_1), B the point (x_2, y_2); and let P be any point of the line drawn through A and B, x and y its coördinates. Draw AC, BD, PM $\parallel$ to OY, and AEF $\parallel$ to OX.

The triangles APF, ABE are similar; therefore,

$$\frac{PF}{AF}=\frac{BE}{AE}.$$

Now, $PF=y-y_1$, $AF=x-x_1$, $BE=y_2-y_1$, $AE=x_2-x_1$.

Therefore, $$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}. \qquad [4]$$

This is the equation required.

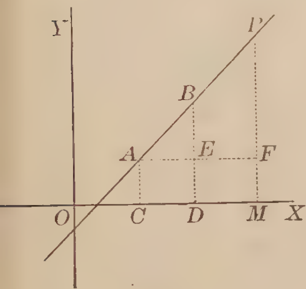


Fig. 16.

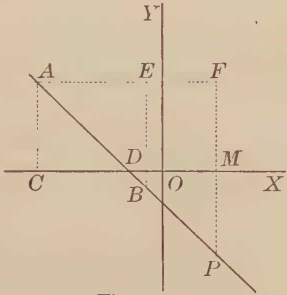


Fig. 17.

It is evident that the angle $FAP=\gamma$. Therefore, each side of equation [4] is equal to $\tan\gamma$ or m. The first side contains the two variables x and y, and the equation tells us that they must vary in such a way that the fraction $\dfrac{y-y_1}{x-x_1}$ remains constant in value and equals m.

NOTE. In Fig. 16 the points A, B, and P are assumed in the first quadrant in order to avoid negative quantities. But the reasoning will lead to equation [4] whatever be the positions of these points. In Fig. 17 the points are in different quadrants. The triangles APF, ABE are to be constructed as shown in the figure. They are similar; and by taking proper account of the algebraic signs of the quantities, we arrive at equation [4], as before. The learner should study this case with care, and should study other cases devised by himself, till he is convinced that equation [4] is perfectly general.

38. *To find the equation of a straight line, given one point (x_1, y_1) in the line and the slope m.*

Let the figure be constructed like Fig. 16, omitting the point B and the line BED. Then it is evident that

$$m = \frac{PF}{AF} = \frac{y - y_1}{x - x_1};$$

whence, $$\boldsymbol{y - y_1 = m(x - x_1).} \qquad [5]$$

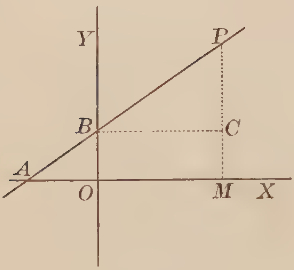


Fig. 18.

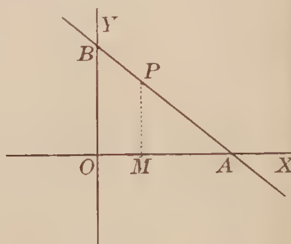


Fig. 19.

39. *To find the equation of a straight line, given the intercept b and the angle γ.*

Let the line cut the axes in the points A and B (Fig. 18). Let P be any point (x, y) in the line. Draw $PM \parallel$ to OY, and $BC \parallel$ to OX.

Then $OB = b$, $PBC = \gamma$, $BC = x$, $PC = y - b$;

therefore, $$m = \frac{y - b}{x};$$

whence, $$\boldsymbol{y = mx + b.} \qquad [6]$$

40. *To find the equation of a straight line, given its intercepts a and b.*

Let the line cut the axes in the points A and B (Fig. 19), and let P be any point (x, y) in the line. Then $OA = a$,

$OB=b$. Draw $PM \perp$ to OX. The triangles PMA, BOA are similar; therefore,

$$\frac{PM}{BO}=\frac{MA}{OA}=\frac{OA-OM}{OA},$$

or
$$\frac{y}{b}=\frac{a-x}{a}=1-\frac{x}{a};$$

whence,
$$\frac{x}{a}+\frac{y}{b}=1. \qquad [7]$$

This is called the **Symmetrical Equation** of the straight line.

41. *To find the equation of a straight line, given its distance p from the origin, and the angle α.*

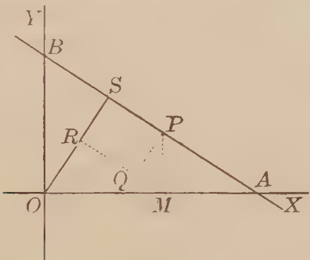


Fig. 20.

Let AB (Fig. 20) be the line, P any point in it. Draw $OS \perp$ to AB, meeting AB in S; $PM \perp$ to OX; $MR \parallel$ to AB, meeting OS in R; and $PQ \perp$ to AB.

Then $\qquad p=OS=OR+QP,\ \alpha=XOS=PMQ.$

Now, $\qquad OR=OM\cos\alpha=x\cos\alpha,$

and $\qquad QP=PM\sin\alpha=y\sin\alpha.$

Therefore, $\qquad OR+QP=p=x\cos\alpha+y\sin\alpha,$

or
$$x\cos\alpha+y\sin\alpha=p. \qquad [8]$$

This is called the **Normal Equation** of the straight line.

The coefficients $\cos\alpha$ and $\sin\alpha$ determine the direction of the line, and p its distance from the origin.

NOTE. Observe that all the equations of the straight line that have been obtained are of the *first* degree. Their differences in form are due to the constants which enter them. The form of each, and the signification of its constants, should be thoroughly fixed in mind.

Exercise 11.

Find the equation of the straight line passing through the two points:

1. $(2, 3)$ and $(4, 5)$.
2. $(4, 5)$ and $(7, 11)$.
3. $(-1, 2)$ and $(3, -2)$.
4. $(-2, -2)$ and $(-3, -3)$.
5. $(4, 0)$ and $(2, 3)$.
6. $(0, 2)$ and $(-3, 0)$.
7. $(2, 5)$ and $(0, 7)$.
8. $(3, 4)$ and $(0, 0)$.
9. $(3, 0)$ and $(0, 0)$.
10. $(3, 4)$ and $(-2, 4)$.
11. $(2, 5)$ and $(-2, -5)$.
12. (m, n) and $(-m, -n)$.

Find the equation of a straight line, given:

13. $(4, 1)$ and $\gamma = 45°$.
14. $(2, 7)$ and $\gamma = 60°$.
15. $(-3, 11)$ and $\gamma = 45°$.
16. $(13, -4)$ and $\gamma = 150°$.
17. $(3, 0)$ and $\gamma = 30°$.
18. $(0, 3)$ and $\gamma = 135°$.
19. $(0, 0)$ and $\gamma = 120°$.
20. $(2, -3)$ and $\gamma = 0°$.
21. $(2, -3)$ and $\gamma = 90°$.
22. $b = 2$, $\gamma = 45°$.
23. $b = 5$, $\gamma = 45°$.
24. $b = -4$, $\gamma = 45°$.
25. $b = -4$, $\gamma = 30°$.
26. $b = -4$, $\gamma = 0°$.
27. $b = -4$, $\gamma = 60°$.
28. $b = -4$, $\gamma = 90°$.
29. $b = -4$, $\gamma = 120°$.
30. $b = -4$, $\gamma = 135°$.
31. $b = -4$, $\gamma = 150°$.
32. $b = -4$, $\gamma = 180°$.
33. $a = 4$, $b = 3$.
34. $a = -6$, $b = 2$.
35. $a = -3$, $b = -3$.
36. $a = 5$, $b = -3$.
37. $a = -10$, $b = 5$.
38. $a = 1$, $b = -1$.
39. $a = n$, $b = -n$.
40. $a = n$, $b = 4n$.
41. $p = 5$, $\alpha = 45°$.
42. $p = 5$, $\alpha = 120°$.
43. $p = 5$, $\alpha = 240°$.
44. $p = 5$, $\alpha = 300°$.

Write the equations of the sides of a triangle:

45. If its vertices are the points $(2, 1)$, $(3, -2)$, $(-4, -1)$.

46. If its vertices are the points $(2, 3)$, $(4, -5)$, $(-3, -6)$.

47. Form the equations of the medians of the triangle described in example 46.

48. The vertices of a quadrilateral are $(0, 0)$, $(1, 5)$, $(7, 0)$, $(4, -9)$. Form the equations of its sides, and also of its diagonals.

Find the equation of a straight line, given:

49. $a = 7\frac{1}{2}$, $\gamma = 30°$.

50. $a = -3$, (x_1, y_1) is $(2, 5)$.

51. $p = 6$, $\gamma = 45°$.

52. $p = 6$, $\gamma = 135°$.

Reduce the following equations to the symmetrical form, and construct each by its intercepts:

53. $3x - 2y + 11 = 0$ and $y = 7x + 1$.

54. $3x + 5y - 13 = 0$ and $4x - y - 2 = 0$.

55. Reduce $Ax + By = C$ to the symmetrical form; also $y = mx + b$. What are the values of a and b in terms of A, B, C, and m?

Reduce the following equations to the form $y = mx + b$, and construct each by its slope, and intercept on the axis of y:

56. $y + 13 = 5x$ and $y + 19 = 7x$.

57. $3x + y + 2 = 0$ and $2y = 3x + 6$.

58. Reduce $Ax + By = C$ to the form $y = mx + b$; also $\frac{x}{a} + \frac{y}{b} = 1$. What are the values of m and b in terms of A, B, C, and a?

59. Find the vertices of the triangle whose sides are the lines $2x + 9y + 17 = 0$, $y = 7x - 38$, $2y - x = 2$.

60. Find the equation of the straight line passing through the origin and the intersection of the lines $3x-2y+4=0$ and $3x+4y=5$. Also find the distance between these two points.

61. What is the equation of the line passing through (x_1, y_1), and equally inclined to the two axes?

62. Find the equations of the diagonals of the parallelogram formed by the lines $x=a$, $x=b$, $y=c$, $y=d$.

63. Show that the lines $y=2x+3$, $y=3x+4$, $y=4x+5$ all pass through one point.

HINT. Find the intersection of two of the lines, and then see if its coördinates will satisfy the equation of the remaining line.

64. The vertices of a triangle are $(0, 0)$, $(x_1, 0)$, (x_2, y_2). Find the equations of its medians, and prove that they meet in one point.

65. What must be the value of m if the line $y=mx$ passes through the point $(1, 4)$?

66. The line $y=mx+3$ passes through the intersection of the lines $y=x+1$ and $y=2x+2$. Determine the value of m.

67. Find the value of b if the line $y=6x+b$ passes through the point $(2, 3)$.

68. What condition must be satisfied if the points (x_1, y_1), (x_2, y_2) (x_3, y_3) lie in one straight line?

HINT. Let equation [4] represent the line through (x_1, y_1) and (x_2, y_2); then (x_3, y_3) must satisfy it.

69. Discuss equation [5] for the following cases: (i) (x_1, y_1) is $(0, 0)$, (ii) $m=0$, (iii) $m=\infty$.

70. Discuss equation [6] for the following cases: (i) $b=0$, (ii) $m=0$, (iii) $m=\infty$, (iv) $m=0$, and $b=0$.

71. Discuss equation [7] for the following cases: (i) $a=b$, (ii) $a=0$, (iii) $a=\infty$, (iv) $b=\infty$.

GENERAL EQUATION OF THE FIRST DEGREE.

42. If any point $P(x_1, y_1)$ is connected with the origin O; then $\frac{x_1}{OP} = \cos XOP$, $\frac{y_1}{OP} = \sin XOP$, and $OP = \sqrt{x_1^2 + y_1^2}$.

Hence, *if any two real quantities are each divided by the square root of the sum of their squares, the quotients are the cosine and sine of some angle.*

43. *The locus of every equation of the first degree in x and y is a straight line.*

Any simple equation in x and y can be reduced to the form

$$Ax + By = C, \qquad [9]$$

in which C is positive or zero.

Dividing both members of [9] by $\sqrt{A^2 + B^2}$, we obtain

$$\frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y = \frac{C}{\sqrt{A^2+B^2}}. \qquad (1)$$

Now, by § 42, the coefficients of x and y in (1) are a set of values of $\cos \alpha$ and $\sin \alpha$, and the second member, being positive, is some value of p (§ 41). Hence (1) is in the normal form, and its locus is some straight line. Whence the proposition.

COR. 1. To reduce any simple equation to the normal form, put it in the form of [9], and divide both members by the square root of the sum of the squares of the coefficients of x and y.

COR. 2. To construct (1), locate the point (A, B), connect it with the origin, and on this line lay off OS equal to the second member of (1); the perpendicular to OS through S is the locus of (1), or [9].

44. The locus of an equation of the first degree in x and y is called a **Locus of the First Order.**

Exercise 12.

Reduce the following equations to the normal form, and thus determine p, or the distance of each locus from the origin:

1. $3x-2y+11=0$.
2. $3x+5y-13=0$.
3. $4x-y-2=0$.
4. $2x+3y=7$.
5. $y+13=5x$.
6. $y+19=7x$.
7. $ex+cy+n=0$.
8. $ny+cx-r=0$.

Reduce the following equations to one of the forms [6], [7], [8], and determine by the signs of the constants which of the four quadrants each locus crosses:

9. $y=\frac{1}{3}x-9$.
10. $3x+2=2y$.
11. $4y=5x-1$.
12. $4y=3x+24$.
13. $5x+3y+15=0$.
14. $5x+4y-20=0$.
15. $y=6x+12$.
16. $y+2=x-4$.
17. $x+\sqrt{3}y+10=0$.
18. $x-\sqrt{3}y-10=0$.

19. Discuss equation [9] for the following cases:

(i) $A=0$. (iv) $A=\infty$. (vii) $A=B, C=0$.

(ii) $B=0$. (v) $A=C=0$. (viii) $A=-B, C=0$.

(iii) $C=0$. (vi) $A=B$.

20. Reduce equation [7] to the form of equation [6], and find the value of m in terms of a and b.

21. What must be the value of C that the line $4x-5y=C$ may pass through the origin? through $(2, 0)$?

22. Determine the values of A, B, and C, that the line $Ax+By=C$ may pass through $(3, 0)$ and $(0, -12)$.

HINT. Since the coördinates of the given points must satisfy the equation, we have the two relations $3A=C$ and $-12B=C$.

23. From [9] deduce [4] by the method used in No. 22.

24. If equations [4] and [9] represent the same line, what are the values of A, B, C, in terms of x_1, y_1, x_2, y_2?

25. In equation [4] find the values of m and b in terms of x_1, y_1, x_2, y_2.

Angles.

45. *To find the angle formed by the lines $y = mx + b$, and $y = m'x + b'$.*

Let AB and CD (Fig. 21) represent the two lines respectively, meeting in the point P.

Let the angle $APC = \phi$; then, by Geometry, $\phi = \gamma - \gamma'$. Whence, by Trigonometry,

$$\tan \phi = \frac{m - m'}{1 + mm'}. \qquad [10]$$

This equation determines the value of ϕ.

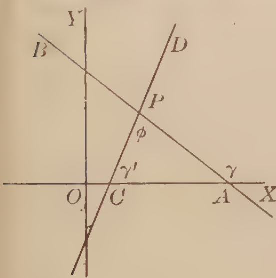


Fig. 21.

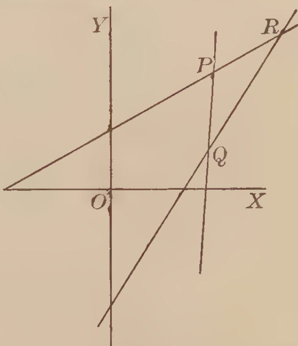


Fig. 22.

Cor. 1. If the lines are parallel, $\tan \phi = 0$; hence, $m = m'$. Conversely, if $m = m'$, $\phi = 0$, and the lines are parallel.

Cor. 2. If the lines are perpendicular, $\tan \phi = \infty$; hence, $1 + mm' = 0$, or $m' = -\dfrac{1}{m}$. Conversely, if $1 + mm' = 0$, $\phi = 90°$, and the lines are perpendicular.

46. *To find the equation of a straight line passing through the point (x_1, y_1) and* (i) *parallel,* (ii) *perpendicular, to the line $y = mx + b$.*

The slope of the required line is m in case (i), and $-\frac{1}{m}$ in case (ii); and in both cases the line passes through a given point (x_1, y_1).

Therefore (§ 38), the required equation is

$$\text{(i)}\ y - y_1 = m(x - x_1),$$

$$\text{(ii)}\ y - y_1 = -\frac{1}{m}(x - x_1).$$

Exercise 13.

Find the equation of the straight line:

1. Passing through $(3, -7)$, and $\parallel$ to the line $y = 3x - 5$.
2. Passing through $(5, 3)$, and $\parallel$ to the line $\frac{1}{3}y - \frac{1}{4}x = 1$.
3. Passing through $(0, 0)$, and $\parallel$ to the line $y - 4x = 10$.
4. Passing through $(5, 8)$, and $\parallel$ to the axis of x.
5. Passing through $(5, 8)$, and $\parallel$ to the axis of y.
6. Passing through $(3, -13)$, and $\perp$ to the line $y = 4x - 7$.
7. Passing through $(2, 9)$, and $\perp$ to the line $7y + 23x - 5 = 0$.
8. Passing through $(0, 0)$, and $\perp$ to the line $x + 2y = 1$.
9. Perpendicular to the line $5x - 7y + 1 = 0$, and erected at the point whose abscissa $= 1$.

47. *To find the equation of a straight line passing through a given point (x_1, y_1), and making a given angle ϕ with a given line $y = mx + b$.*

Let the required equation be

$$y - y_1 = m'(x - x_1),$$

where m' is not yet determined.

Since the required line may lie either as PQ or PR (Fig. 22), we shall have (§ 45),

$$\tan\phi = \frac{m'-m}{1+mm'} \text{ or } \frac{m-m'}{1+mm'}.$$

Hence,
$$m' = \frac{m \pm \tan\phi}{1 \mp m\tan\phi},$$

and the required equation is

$$y - y_1 = \frac{m \pm \tan\phi}{1 \mp m\tan\phi}(x - x_1), \qquad [11]$$

and (as Fig. 22 shows) there are in general two straight lines satisfying the given conditions.

Exercise 14.

1. Find the angle formed by the lines $x + 2y + 1 = 0$ and $x - 3y - 4 = 0$.

The two slopes are $-\frac{1}{2}$ and $\frac{1}{3}$. If we put $m = -\frac{1}{2}$, $m' = \frac{1}{3}$, we obtain $\tan\phi = -1$, $\phi = 135°$. If we put $m = \frac{1}{3}$, $m' = -\frac{1}{2}$, we get $\tan\phi = 1$, $\phi = 45°$. Show that both these results are correct.

Find the tangent of the angle formed by the lines:

2. $3x - 4y = 7$ and $2x - y = 3$.
3. $2x + 3y + 4 = 0$ and $3x + 4y + 5 = 0$.
4. $y - nx = 1$ and $2(y - 1) = nx$.

Find the angle formed by the lines:

5. $x + y = 1$ and $y = x + 4$.
6. $y + 3 = 2x$ and $y + 3x = 2$.
7. $2x + 3y + 7 = 0$ and $3x - 2y + 4 = 0$.
8. $6x = 2y + 3$ and $y - 3x = 10$.
9. $x + 3 = 0$ and $y - \sqrt{3}x + 4 = 0$.

10. Discuss equation [11] for the cases when $\phi = 0°$ and $\phi = 90°$.

NOTE. The learner should solve the next four exercises directly without using equation [11]; then verify the result by means of [11].

Find the equation of a straight line:

11. Passing through the point (3, 5), and making the angle 45° with the line $2x - 3y + 5 = 0$.

12. Passing through the point $(-2, 1)$, and making the angle 45° with the line $2y = 6 - 3x$.

13. Passing through that point of the line $y = 2x - 1$ for which $x = 2$, and making the angle 30° with the same line.

14. Passing through (1, 3), and making the angle 30° with the line $x - 2y + 1 = 0$.

15. Prove that the lines represented by the equations

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0$$

are parallel if $AB' = A'B$; perpendicular, if $AA' = -BB'$.

16. Given the equation $3x + 4y + 6 = 0$; show that the general equations representing (i) all parallels and (ii) all perpendiculars to the given line are

$$\text{(i)} \quad 3x + 4y + K = 0.$$
$$\text{(ii)} \quad 4x - 3y + K = 0.$$

17. Deduce the following equations for lines passing through (x_1, y_1) and (i) parallel, (ii) perpendicular, to the line $y = mx + b$.

$$\text{(i)} \quad y - mx = y_1 - mx_1.$$
$$\text{(ii)} \quad my + x = my_1 + x_1.$$

18. Write the equations of three lines parallel, and three lines perpendicular, to the line $2x + 3y + 1 = 0$.

19. Among the following lines select parallel lines; perpendicular lines; lines neither parallel nor perpendicular:

(i) $2x+3y-1=0$. (v) $x-y=2$.
(ii) $3x-2y=20$. (vi) $5(x+y)-11=0$.
(iii) $4x+6y=0$. (vii) $x=8$.
(iv) $12x=8y+7$. (viii) $y+10=0$.

20. Prove that the angle ϕ, between the lines

$$Ax+By+C=0 \text{ and } A'x+B'y+C'=0,$$

is determined by the equation

$$\tan\phi=\frac{A'B-AB'}{AA'+BB'}.$$

21. From the preceding equation deduce the conditions of parallel lines and perpendicular lines given in No. 15.

Find the equation of the straight line:

22. Parallel to $2x+3y+6=0$, and passing through $(5, 7)$.

23. Parallel to $2x+y-1=0$, and passing through the intersection of $3x+2y-59=0$ and $5x-7y+6=0$.

24. Parallel to the line joining $(-2, 7)$ and $(-4, -5)$, and passing through $(5, 3)$.

25. Parallel to $y=mx+b$, and at a distance d from the origin.

26. Perpendicular to $Ax+By+C=0$, and cutting an intercept b on the axis of y.

27. Perpendicular to $\frac{x}{a}+\frac{y}{b}=1$, and passing through (a, b).

28. Making the angle 45° with $\frac{x}{a}+\frac{y}{b}=1$, and passing through $(a, 0)$.

29. Show that the triangle whose vertices are the points $(2, 1)$, $(3, -2)$, $(-4, -1)$ is a right triangle.

30. The vertices of a triangle are $(-1, -1)$, $(-3, 5)$, $(7, 11)$. Find the equations of its altitudes. Prove that the altitudes meet in one point.

31. Find the equation of the perpendicular erected at the middle point of the line joining $(5, 2)$ to the intersection of $x + 2y - 11 = 0$ and $9x - 2y - 59 = 0$.

32. Find the equations of the perpendiculars erected at the middle points of the sides of the triangle whose vertices are $(5, -7)$, $(1, 11)$, $(-4, 13)$. Prove that these perpendiculars meet in one point.

33. The equations of the sides of a triangle are

$$x + y + 1 = 0,\ 3x + 5y + 11 = 0,\ x + 2y + 4 = 0.$$

Find (i) the equations of the perpendiculars erected at the middle points of the sides; (ii) the coördinates of their common point of intersection; (iii) the distance of this point from the vertices of the triangle.

34. Show that the straight line passing through (a, b) and (c, d) is perpendicular to the straight line passing through $(b, -a)$ and $(d, -c)$.

35. What is the equation of the straight line passing through (x_1, y_1), and making an angle ϕ with the line $Ax + By + C = 0$?

Distances.

48. *Find the distance from the point* $(-4, 1)$ *to the line* $3x - 4y + 1 = 0$. *Ans.* 3.

The required distance is the length of the perpendicular from the given point to the given line. The method that first suggests itself is to form the equation of this perpendicular, find its intersection with the given line, and compute the distance from this intersection to the given point.

Let this method be followed in solving the above problem and the first five problems of Exercise 15.

49. *To find the distance from the point (x_1, y_1) to the line*

$$x \cos \alpha + y \sin \alpha = p.$$

Let the line $$x \cos \alpha + y \sin \alpha = p', \qquad (1)$$
which is evidently parallel to the given line, pass through the given point (x_1, y_1); then we have

$$x_1 \cos \alpha + y_1 \sin \alpha = p'.$$

Therefore, $$x_1 \cos \alpha + y_1 \sin \alpha - p = p - p'.$$

But $p' - p$ equals numerically the required distance. Therefore, the distance from the point (x_1, y_1) to the line $x \cos \alpha + y \sin \alpha = p$ is obtained by substituting x_1 for x and y_1 for y in the expression $x \cos \alpha + y \sin \alpha - p$.

Cor. 1. The distance as obtained from the formula will evidently be *positive* or *negative* according as the point and origin are on opposite sides of the line, or on the same side.

Cor. 2. If the equation of the line is

$$Ax + By = C,$$

and d denotes the distance from (x_1, y_1) to the line; then, evidently,

$$d = \frac{Ax_1 + By_1 - C}{\sqrt{A^2 + B^2}}. \qquad [12]$$

Hence, to find the distance from the point (x_1, y_1) to the line $Ax + By = C$, write x_1 for x, and y_1 for y in the expression $Ax + By - C$, and divide the result by $\sqrt{A^2 + B^2}$.

For example, let $(-1, 3)$ be the point, and $2x + 4 = 3y$ the equation of the line.

Putting this equation in the form of [9], we have

$$-2x + 3y = 4.$$

Whence, $$d = \frac{-2\,(-1) + 3 \times 3 - 4}{\sqrt{(-2)^2 + 3^2}} = + \tfrac{7}{13}\sqrt{13}.$$

Hence, the point and origin are on opposite sides of the line. If only the length of d is sought, its sign may be neglected.

Exercise 15.

1. Find the distance from $(1, 13)$ to the line $3x = y - 5$.

2. Find the distance from $(8, 4)$ to the line $y = 2x - 16$.

3. Find the distance from the origin to the line $3x + 4y = 20$.

4. Find the distance from $(2, 3)$ to the line $2x + y - 4 = 0$.

5. Find the distance from $(3, 3)$ to the line $y = 4x - 9$.

6. Prove that the distance from the point (x_1, y_1) to the line $y = mx + b$ is

$$d = \pm \frac{y_1 - mx_1 - b}{\sqrt{1 + m^2}},$$

the upper or lower sign being used according as b is positive or negative. Express this result in the form of a rule for practice.

7. Find the distances from the line $3x + 4y + 15 = 0$ to the following points: $(3, 0)$, $(3, -1)$, $(3, -2)$, $(3 - 3)$, $(3, -4)$, $(3, -5)$, $(3, -6)$, $(3, -7)$, $(0, 0)$, $(-1, 0)$, $(-2, 0)$, $(-3, 0)$, $(-4, 0)$, $(-5, 0)$, $(-6, 0)$.

8. Find the distances from $(1, 3)$ to the following lines:

$3x + 4y + 15 = 0.$	$3x + 4y - 5 = 0.$
$3x + 4y + 10 = 0.$	$3x + 4y - 10 = 0.$
$3x + 4y + 5 = 0.$	$3x + 4y - 15 = 0.$
$3x + 4y = 0.$	$3x + 4y - 20 = 0.$

Find the following distances:

9. From the point $(2, -5)$ to the line $y - 3x = 7$.

10. From the point $(4, 5)$ to the line $4y + 5x = 20$.

11. From the point $(2, 3)$ to the line $x + y = 1$.

12. From the point $(0, 1)$ to the line $3x - 3y = 1$.

13. From the point $(-1, 3)$ to the line $3x + 4y + 2 = 0$.

14. From the origin to the line $3x+2y-6=0$.

15. From the point $(2, -7)$ to the line joining $(-4, 1)$ and $(3, 2)$.

16. From the line $y=7x$ to the intersection of the lines $y=3x-4$ and $y=5x+2$.

17. From the origin to the line $a(x-a)+b(y-b)=0$.

18. From the points (a, b) and $(-a, -b)$ to the line

$$\frac{x}{a}+\frac{y}{b}=1.$$

19. From the point (a, b) to the line $ax+by=0$.

20. From the point (h, k) to the line $Ax+By+C=D$.

Find the distance between the two parallels:

21. $3x+4y+15=0$ and $3x+4y+5=0$.

22. $3x+4y+15=0$ and $3x+4y-5=0$.

23. $Ax+By=C$ and $Ax+By=C'$.

24. $Ax+By=C$ and $-Ax-By=C'$.

25. $y=5x-7$ and $y=5x+3$.

26. $\frac{x}{a}+\frac{y}{b}=2$ and $\frac{x}{a}+\frac{y}{b}=\frac{1}{2}$.

27. Show that the locus of a point which is equidistant from the lines $3x+4y-12=0$ and $4x+3y-24=0$ consists of two straight lines. Find their equations, and draw a figure representing the four lines.

28. Show that the locus of a point which so moves that the sum of its distances from two given straight lines is constant is a straight line.

Areas.

50. *To find the area of a triangle, having given its vertices.*

Solution I. Let PQR (Fig. 23) be the given triangle, and let the vertices of PQR be (x_1, y_1), (x_2, y_2), (x_3, y_3), respectively. Drop the perpendiculars PM, QN, RL; then

$$\text{area } PQR = PQNM + RLNQ - PMLR.$$

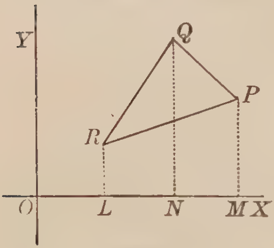


Fig 23.

By Geometry,

$$PQNM = \tfrac{1}{2}\, NM(MP + NQ)$$
$$= \tfrac{1}{2}(x_1 - x_2)(y_1 + y_2).$$

Similarly,

$$RLNQ = \tfrac{1}{2}(x_2 - x_3)(y_3 + y_2).$$
$$PMLR = \tfrac{1}{2}(x_1 - x_3)(y_3 + y_1).$$

Substituting these values, we have

$$\text{area } PQR = \tfrac{1}{2}[(x_1 - x_2)(y_2 + y_1) + (x_2 - x_3)(y_3 + y_2) - (x_1 - x_3)(y_3 + y_1)]$$
$$= \tfrac{1}{2}[-x_2y_1 + x_1y_2 - x_3y_2 + x_2y_3 - x_1y_3 + x_3y_1].$$

$$\therefore \textbf{area} = \tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]. \qquad [13]$$

Solution II. Since the area of a triangle is equal to one half the product of its base and its altitude, this problem may be solved as follows:

(i) Find the length of any side as base.
(ii) Find the equation of the base.
(iii) Find the distance of the base from the opposite vertex.
(iv) Multiply this distance by one half the base.

Exercise 16.

Find the area of the triangle whose vertices are the points:

1. $(0, 0), (1, 2), (2, 1)$.
2. $(3, 4), (-3, -4), (0, 4)$.
3. $(2, 3), (4, -5), (-3, -6)$.
4. $(8, 3), (-2, 3), (4, -5)$.
5. $(a, 0), (-a, 0), (0, b)$.

6. Compare the formula for the area of a triangle with the result obtained by solving No. 68, p. 42. What, then, is the geometric meaning of that result?

Find the area of the figure having for vertices the points:

7. $(3, 5), (7, 11), (9, 1)$.

8. $(3, -2), (5, 4), (-7, 3)$.

9. $(-1, 2), (4, 4), (6, -3)$.

10. $(0, 0), (x_1, y_1), (x_2, y_2)$.

11. $(2, -5), (2, 8), (-2, -5)$.

12. $(10, 5), (-2, 5), (-5, -3), (7, -3)$.

13. $(0, 0), (5, 0), (9, 11), (0, 3)$.

14. $(a, 1), (0, b), (c, 1)$.

15. $(a, b), (b, a), (c, c)$.

16. $(a, b), (b, a), (c, -c)$.

17. Find the angles and the area of the triangle whose vertices are $(3, 0)$, $(0, 3\sqrt{3})$, $(6, 3\sqrt{3})$.

What is the area contained by the lines:

18. $x=0,\quad y=0,\quad 5x+4y=20$?

19. $x+y=1,\quad x-y=0,\quad y=0$?

20. $x+2y=5,\quad 2x+y=7,\quad y=x+1$?

21. $x+y=0,\quad x=y,\quad y=3a$?

22. $y=3x,\quad y=7x,\quad y=c$?

23. $x=0,\quad y=0,\quad x-4=0,\quad y+6=0$?

24. $3x+y+4=0,\ 3x-5y+34=0,\ 3x-2y+1=0$?

25. $x-5y+13=0,\ 5x+7y+1=0,\ 3x+y-9=0$?

26. $x-y=0,\quad x+y=0,\quad x-y=a,\quad x+y=b$?

Find the area contained by the lines:

27. $x=0,\quad y=0,\quad y=mx+b.$

28. $x=0,\quad y=0,\quad \frac{x}{a}+\frac{y}{b}=1.$

29. $x=0,\quad y=0,\quad Ax+By+C=0.$

30. $y=3x-9,\quad y=3x+5,\quad 2y=x-6,\quad 2y=x+14.$

31. What is the area of the triangle formed by drawing straight lines from the point (2, 11) to the points in the line $y=5x-6$ for which $x=4$, $x=7$?

Exercise 17. (Review.)

1. Deduce equation [7], p. 39, from equation [6].

2. The equation $y=mx+b$ is not so general as the equation $Ax+By+C=0$, because it cannot represent a line parallel to the axis of y. Explain more fully.

Determine for the following lines the values of a, b, γ, p, and α:

3. $x=2.$

4. $x=y.$

5. $y+1=\sqrt{3}(x+2).$

6. $x+\sqrt{3}y=2.$

7. $x-\sqrt{3}y=2.$

8. $\sqrt{3}x-y=2.$

9. Find the equations of the diagonals of the figure formed by the lines $3x-y+9=0$, $3x=y-1$, $5x+3y=18$, $5x+3y=2$. What kind of quadrilateral is it? Why?

10. Find the distance between the parallels $9x=y+1$ and $9x=y-7$.

11. The vertices of a quadrilateral are (3, 12), (7, 9), (2, −3), (−2, 0). Find the equations of its sides and its area.

12. The vertices of a quadrilateral are (6, −4), (4, 4), (−4, 2), (−8, −6). Prove that the lines joining the middle points of adjacent sides form a parallelogram. Find the area of this parallelogram.

Find the equation of a line passing through $(3, 4)$, and also:

13. Perpendicular to the axis of x.

14. Making the angle $45°$ with the axis of x.

15. Parallel to the line $5x + 6y + 8 = 0$.

16. Intercepting on the axis of y the distance -10.

17. Passing through the point halfway between $(1, -4)$ and $(-5, 4)$.

18. Perpendicular to the line joining $(3, 4)$ and $(-1, 0)$.

Find the equations of the following lines:

19. A line parallel to the line joining (x_1, y_1) and (x_2, y_2), and passing through (x_3, y_3).

20. The lines passing through $(8, 3)$, $(4, 3)$, $(-5, -2)$.

21. A line passing through the intersection of the lines $2x + 5y + 8 = 0$ and $3x - 4y - 7 = 0$, and $\perp$ to the latter line.

22. A line $\perp$ to the line $4x - y = 0$, and passing through that point of the given line whose abscissa is 2.

23. A line $\parallel$ to the line $3x + 4y = 0$, and passing through the intersection of the lines $x - 2y - a = 0$ and $x + 3y - 2a = 0$.

24. A line through $(4, 3)$, such that the given point bisects the portion contained between the axes.

25. A line through (x_1, y_1), such that the given point bisects the portion contained between the axes.

26. A line through $(4, 3)$, and forming with the axes in the second quadrant a triangle whose area is 8.

27. A line through $(4, 3)$, and forming with the axes in the fourth quadrant a triangle whose area is 8.

28. A line through $(-4, 3)$, such that the portion between the axes is divided by the given point in the ratio $5:3$.

29. A line dividing the distance between $(-3, 7)$ and $(5, -4)$ in the ratio $4:7$, and $\perp$ to the line joining these points.

30. The two lines through $(3, 5)$ making the angle $45°$ with the line $2x - 3y - 7 = 0$.

31. The two lines through $(7, -5)$ that make the angle $45°$ with the line $6x - 2y + 3 = 0$.

32. The line making the angle $45°$ with the line joining $(7, -1)$ and $(-3, 5)$, and intercepting the distance 5 on the axis of x.

33. The two lines that pass through the origin and trisect the portion of the line $x + y = 1$ included between the axes.

34. The two lines $\parallel$ to the line $4x + 5y + 11 = 0$, at the distance 3 from it.

35. The bisectors of the angles contained between the lines $y = 2x + 4$ and $-y = 3x + 6$.

HINT. Every point in the bisector of an angle is equidistant from the sides of the angle.

36. The bisectors of the angles contained between the lines $2x - 5y = 0$ and $4x + 3y = 12$.

37. The two lines that pass through $(3, 12)$, and whose distance from $(7, 2)$ is equal to $\sqrt{58}$.

38. The two lines that pass through $(-2, 5)$, and are each equidistant from $(3, -7)$ and $(-4, 1)$.

Find the angle contained between the lines:

39. $y + 3 = 2x$ and $y + 3x = 2$.

40. $y = 5x - 7$ and $5y + x - 3 = 0$.

Find the distance:

41. From the intersection of the lines $3x+2y+4=0$, $2x+5y+8=0$ to the line $y=5x+6$.

42. From the point (h, k) to the line $\frac{x}{a}+\frac{y}{b}=1$.

43. From the origin to the line $hx+ky=c^2$.

44. From the point $(a, 0)$ to the line $y=mx+\frac{a}{m}$.

Find the area included by the following lines:

45. $x=y$, $x+y=0$, $x=c$.

46. $x+y=k$, $2x=y+k$, $2y=x+k$.

47. $\frac{x}{a}+\frac{y}{b}=1$, $y=2x+b$, $x=2y+a$.

48. $y=4x+7$ and the lines that join the origin to those points of the given line whose ordinates are -1 and 19.

49. The lines joining the middle points of the sides of the triangle formed by the lines $x-5y+11=0$, $11x+6y-1=0$, $x+y+4=0$.

50. Find the area of the quadrilateral whose vertices are $(0, 0)$, $(0, 5)$, $(11, 9)$, $(7, 0)$.

51. What point in the line $5x-4y-28=0$ is equidistant from the points $(1, 5)$ and $(7, -3)$?

52. Prove that the diagonals of a square are perpendicular to each other.

53. Prove that the line joining the middle points of two sides of a triangle is parallel to the third side.

54. What is the geometric meaning of the equation $xy=0$?

55. Show that the three points $(3a, 0)$, $(0, 3b)$, $(a, 2b)$ are in a straight line.

56. Show that the three lines $5x + 3y - 7 = 0$, $3x - 4y - 10 = 0$, and $x + 2y = 0$ meet in a point.

57. What must be the value of a in order that the three lines $3x + y - 2 = 0$, $2x - y - 3 = 0$, and $ax + 2y - 3 = 0$ may meet in a point?

What straight lines are represented by the equations:

58. $x^2 + (a - b)x - ab = 0$?

59. $xy + bx + ay + ab = 0$?

60. $x^2y = xy^2$?

61. $14x^2 - 5xy - y^2 = 0$?

In the following exercises prove that the locus of the point is a straight line, and obtain its equation:

62. The locus of the vertex of a triangle having the base and the area constant.

63. The locus of a point equidistant from the points (x_1, y_1) and (x_2, y_2).

64. The locus of a point at the distance d from the line $Ax + By + C = 0$.

65. The locus of a point so moving that the sum of its distances from the axes shall be constant and equal to k.

66. The locus of a point so moving that the sum of its distances from the lines $Ax + By + C = 0$, $A'x + B'y + C' = 0$ shall be constant and equal to k.

67. The locus of the vertex of a triangle, having given the base and the difference of the squares of the other sides.

SUPPLEMENTARY PROPOSITIONS.

LINES PASSING THROUGH ONE POINT.

51. *If $S=0$, $S'=0$ represent the equations of any two loci with the terms all transposed to the left-hand side, and k denotes an arbitrary constant, then the locus represented by the equation $S+kS'=0$ passes through every point common to the two given loci.*

For if any coördinates satisfy the equation $S=0$, and also satisfy the equation $S'=0$, they must likewise satisfy the equation $S+kS'=0$.

For what values of k will the equation $S+kS'=0$ represent the lines $S=0$ and $S'=0$, respectively?

52. *Find the equation of the line joining the point* (3, 4) *to the intersection of the lines*

$$3x-2y+17=0 \text{ and } x+4y-27=0.$$

The method of solving this question that first suggests itself is to find the intersection of the given lines and then apply equation [4], p. 37.

Another method, almost equally obvious, is to employ equation [5], which gives at once

$$y-4=m(x-3),$$

and then determine m by substituting for x and y the coördinates of the intersection of the given lines.

The following method, founded on the principle stated in § 51, is, however, sometimes preferable on account of its generality and because it saves the labor of solving the given equations. According to this principle, the required equation may be immediately written in the form

$$3x-2y+17+k(x+4y-27)=0.$$

And since the line passes through (3, 4), we must have

$$9-8+17+k(3+16-27)=0,$$

whence, $k=\frac{9}{4}$.

Therefore, $12x-8y+68+9x+36y-243=0$,

or $3x+4y-25=0$.

This is the equation of the required line

53. *If the equations of three straight lines are*

$$\begin{aligned} Ax+By+C&=0,\\ A'x+B'y+C'&=0,\\ A''x+B''y+C''&=0,\end{aligned}$$

and we can find three constants, l, m, n, so that the relation $l(Ax+By+C)+m(A'x+B'y+C')+n(A''x+B''y+C'')=0$ is identically true; that is, true for all values of x and y, then the three lines meet in a point.

For if the coördinates of any point satisfy any two of the equations, then the above relation shows that they will also satisfy the third equation.

54. *To find the equation of the bisector of the angle between the two lines*

$$x\cos\alpha+y\sin\alpha=p,$$

and $$x\cos\alpha'+y\sin\alpha'=p'.$$

There are evidently two bisectors: one bisecting the angle in which the origin lies; the other bisecting the supplementary angle.

Now, every point in either bisector is equally distant from the sides of the angle. Let (x, y) be any point in the bisector of the angle that includes the origin; then (§ 49)

$$x\cos\alpha+y\sin\alpha-p=x\cos\alpha'+y\sin\alpha'-p'. \qquad (1)$$

Since (x, y) is any point in this bisector, (1) is its equation.

The equation of the other bisector is

$$x\cos\alpha + y\sin\alpha - p = -(x\cos\alpha' + y\sin\alpha' - p'). \quad (2)$$

To distinguish equations (1) and (2) we note that in the first the constant terms in the two members have like signs; while in the second the constant terms have unlike signs.

Cor. 1. If the equations of the lines are in the form

$$Ax + By = C,\ A'x + B'y = C',$$

the equations of the bisectors are evidently

$$\frac{Ax + By - C}{\sqrt{A^2 + B^2}} = \pm \frac{A'x + B'y - C'}{\sqrt{A'^2 + B'^2}}. \qquad [14]$$

Equation [14] represents the bisector of the angle in which the origin lies, or of its supplementary angle, *according as we take the upper or lower sign.*

For example, let the equations of the lines be

$$2x = 4y + 9, \text{ and } 5y = 3x - 7.$$

Putting these equations in the form of [9], we have

$$2x - 4y = 9, \text{ and } 3x - 5y = 7.$$

Hence, the equations of the bisectors of their included angles are

$$\frac{2x - 4y - 9}{\sqrt{20}} = \pm \frac{3x - 5y - 7}{\sqrt{34}},$$

in which the upper sign gives the equation of the bisector of the angle in which the origin lies.

Cor. 2. If $S = 0$ and $S' = 0$ represent two simple equations in the normal form, with the terms all transposed to the first members, then the equations of the bisectors of their included angles may be written

$$S = \pm S', \text{ or } S \mp S' = 0.$$

Exercise 18.

Find the equation of the line passing through the intersection of the lines $3x-2y+17=0$, $x+4y-27=0$, and:

1. Passing also through the origin.
2. Parallel to the line $x+2y+3=0$.
3. Perpendicular to the line $6x-5y=0$.
4. Equally inclined to the two axes.

5. Find the equation of the line parallel to the line $x=y$, and passing through the intersection of the lines

$$y=2x+1 \text{ and } y+3x=11.$$

6. Find the equation of the straight line joining $(2, 3)$ to the intersection of the lines

$$2x+3y+1=0 \text{ and } 3x-4y=5.$$

7. Find the equation of the straight line joining $(0, 0)$ to the intersection of the lines

$$5x-2y+3=0 \text{ and } 13x+y=1.$$

8. Find the equation of the straight line joining $(1, 11)$ to the intersection of the lines

$$2x+5y-8=0 \text{ and } 3x-4y=8.$$

Find the equation of the straight line passing through the intersection of the lines $Ax+By+C=0$ and $A'x+B'y+C'=0$, and also:

9. Passing through the origin.
10. Drawn parallel to the axis of x.
11. Passing through the point (x_1, y_1).

12. Find the equation of the straight line passing through the intersection of $5x-4y+3=0$ and $7x+11y-1=0$, and cutting on the axis of y an intercept equal to 6.

13. Find the equation of the straight line passing through the intersection of $y = 7x - 4$ and $y = -2x + 5$, and forming with the axis of x the angle $60°$.

14. The distance of a straight line from the origin is 5; and it passes through the intersection of the lines $3x - 2y + 11 = 0$ and $6x + 7y - 55 = 0$. What is its equation?

15. What is the equation of the straight line passing through the intersection of $bx + ay = ab$ and $y = mx$, and perpendicular to the former line?

Prove that the following lines are concurrent (or pass through one point):

16. $y = 2x + 1, \quad y = x + 3, \quad y = -5x + 15.$

17. $4x - 2y - 3 = 0, \quad 3x - y + \frac{1}{2} = 0, \quad 5x - 2y - 1 = 0.$

18. $2x - y = 5, \quad 3x - y = 6, \quad 4x - y = 7.$

19. What is the value of m if the lines

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{b} + \frac{y}{a} = 1, \quad y = mx$$

meet in one point?

20. When do the straight lines $y = mx + b$, $y = m'x + b'$, $y = m''x + b''$ pass through one point?

21. Prove that the three altitudes of a triangle meet in one point.

22. Prove that the perpendiculars erected at the middle points of the sides of a triangle meet in one point.

23. Prove that the three medians of a triangle meet in one point. Show also that this point is one of the two points of trisection for each median.

24. Prove that the bisectors of the three angles of a triangle meet in one point.

25. The vertices of a triangle are $(2, 1)$, $(3, -2)$, $(-4, -1)$. Find the lengths of its altitudes. Is the origin within or without the triangle?

26. The equations of the sides of a triangle are

$$3x + y + 4 = 0, \quad 3x - 5y + 34 = 0, \quad 3x - 2y + 1 = 0.$$

Find the lengths of its altitudes.

What are the equations of the lines bisecting the angles between the lines :

27. $3x - 4y + 7 = 0$ and $4x - 3y + 17 = 0$?

28. $3x + 4y - 9 = 0$ and $12x + 5y - 3 = 0$?

29. $y = 2x - 4$ and $2y = x + 10$?

30. $x + y = 2$ and $x - y = 0$?

31. $y = mx + b$ and $y = m'x + b'$?

32. Prove that the bisectors of the two supplementary angles formed by two intersecting lines are perpendicular to each other.

Equations representing Straight Lines.

55. *A homogeneous equation of the nth degree represents n straight lines through the origin.*

Let the equation be

$$Ax^n + Bx^{n-1}y + Cx^{n-2}y^2 + \cdots + Ky^n = 0.$$

Dividing by Ay^n, we have

$$\left(\frac{x}{y}\right)^n + \frac{B}{A}\left(\frac{x}{y}\right)^{n-1} + \frac{C}{A}\left(\frac{x}{y}\right)^{n-2} + \cdots + \frac{K}{A} = 0.$$

If $r_1, r_2, r_3, \ldots r_n$ denote the roots of this equation, then the equation, resolved into its factors, becomes

$$\left(\frac{x}{y} - r_1\right)\left(\frac{x}{y} - r_2\right)\left(\frac{x}{y} - r_3\right)\cdots\left(\frac{x}{y} - r_n\right) = 0,$$

and therefore is satisfied when any one of these factors is zero, and in no other cases.

Therefore, the locus of the equation consists of the n straight lines

$$x-r_1y=0,\ x-r_2y=0,\ \ldots\ldots,\ x-r_ny=0;$$

and these lines evidently all pass through the origin.

56. *To find the angle between the two straight lines represented by the equation $Ax^2+Cxy+By^2=0$.*

Solving the equation as a quadratic in x, we obtain

$$2Ax+(C\pm\sqrt{C^2-4AB})\,y=0.$$

Hence, the slopes of the two lines are

$$m=\frac{2A}{-C-\sqrt{C^2-4AB}},\quad m'=\frac{2A}{-C+\sqrt{C^2-4AB}}.$$

Therefore,

$$m-m'=\frac{\sqrt{C^2-4AB}}{B},\quad mm'=\frac{A}{B};$$

and (equation [10], p. 45),

$$\tan\phi=\frac{m-m'}{1+mm'}=\frac{\sqrt{C^2-4AB}}{A+B}.$$

57. *To find the condition that the general equation of the second degree may represent two straight lines.*

We may write the most general form of the equation of the second degree as follows:

$$Ax^2+By^2+Cxy+Dx+Ey+F=0. \qquad (1)$$

That this equation may represent two straight lines, its first member must be the product of two linear factors in x and y; that is, the equation can be written in the form

$$(lx+my+n)\,(px+qy+r)=0. \qquad (2)$$

Equating coefficients in (1) and (2), we obtain

$$lp=A,\qquad mq=B,\qquad nr=F.$$
$$lq+mp=C,\quad lr+np=D,\quad mr+nq=E.$$

The product of C, D, and E is

$$\begin{aligned} CDE &= 2lmnpqr + lp(n^2q^2 + m^2r^2) + mq(l^2r^2 + n^2p^2) \\ &\qquad + nr(l^2q^2 + m^2p^2) \\ &= 2ABF + A(E^2 - 2BF) + B(D^2 - 2AF) \\ &\qquad + F(C^2 - 2AB). \end{aligned}$$

Hence, the required condition is

$$F(C^2 - 4AB) + AE^2 + BD^2 - CDE = 0. \qquad (3)$$

Exercise 19.

1. Describe the position of the two straight lines represented by the equation $Ax^2 + Cxy + By^2 + Dx + Ey + F = 0$, when (i) $A = C = D = 0$, (ii) $B = C = E = 0$.

2. When will the equation $axy + bx + cy + d = 0$ represent two straight lines?

3. Find the conditions that the straight lines represented by the equation $Ax^2 + Cxy + By^2 = 0$ may be real; imaginary; coincident; perpendicular to each other.

4. Show that the two straight lines $x^2 - 2xy \sec\theta + y^2 = 0$ make the angle θ with each other.

Show that the following equations represent straight lines, and find their separate equations:

5. $x^2 - 2xy - 3y^2 + 2x - 2y + 1 = 0$.

6. $x^2 - 4xy + 5y^2 - 6y + 9 = 0$.

7. $x^2 - 4xy + 3y^2 + 6y - 9 = 0$.

8. Show that the equation $x^2 + xy - 6y^2 + 7x + 31y - 18 = 0$ represents two straight lines, and find the angle between them.

Determine the values of K for which the following equations will represent in each case a pair of straight lines. Are the lines real or imaginary?

9. $12x^2 - 10xy + 2y^2 + 11x - 5y + K = 0$.

10. $12x^2 + Kxy + 2y^2 + 11x - 5y + 2 = 0.$

11. $12x^2 + 36xy + Ky^2 + 6x + 6y + 3 = 0.$

12. For what value of K does the equation $Kxy + 5x + 3y + 2 = 0$ represent two straight lines?

Problems on Loci involving Three Variables.

58. *A trapezoid is formed by drawing a line parallel to the base of a given triangle. Find the locus of the intersection of its diagonals.*

If ABC is the given triangle, and we choose for axes the base AB and the altitude CO, the vertices A, B, C may be represented in general by $(a, 0)$, $(b, 0)$, $(0, c)$, respectively. The equations of AC and BC are, respectively,

$$\frac{x}{a} + \frac{y}{c} = 1 \text{ and } \frac{x}{b} + \frac{y}{c} = 1.$$

Let $y = m$ be the equation of the line parallel to the base, and let it cut AC in D, BC in E; then the coördinates of D and E, respectively, are

$$\left(\frac{ac - am}{c}, m\right) \text{ and } \left(\frac{bc - bm}{c}, m\right).$$

Hence, the equation of the diagonal BD is

$$\frac{y}{x - b} = \frac{cm}{ac - am - bc}, \tag{1}$$

and the equation of the diagonal AE is

$$\frac{y}{x - a} = \frac{cm}{bc - bm - ac}. \tag{2}$$

If P is the intersection of the diagonals, then the coördinates x and y of the point P must satisfy both (1) and (2); by solving these equations, therefore, we obtain *for any particular value of* m the coördinates of the point P. But what we want is the algebraic relation that is satisfied

by the coördinates of P, *whatever the value of m may be.* To find this, we have only to eliminate m from equations (1) and (2). By doing this we obtain

$$2cx + (a+b)y = (a+b)c,$$

or

$$\frac{x}{\frac{1}{2}(a+b)} + \frac{y}{c} = 1.$$

We see from the form of this equation that the required locus is the line that joins C to the middle point of AB.

REMARK. The above solution should be studied till it is understood. In problems on loci it is often necessary to obtain relations that involve not only the x and y of a point of the locus which we are seeking, but also *some third variable* (as m in the above example).

In such cases we must obtain *two* equations that involve x and y and this third variable, and then eliminate the third variable; the resulting equation will be the equation of the locus required.

Exercise 20.

1. Through a fixed point O any straight line is drawn, meeting two given parallel straight lines in P and Q; through P and Q straight lines are drawn in fixed directions, meeting in R. Prove that the locus of R is a straight line, and find its equation.

2. The hypotenuse of a right triangle slides between the axes of x and y, its ends always touching the axes. Find the locus of the vertex of the right angle.

3. Given two fixed points, A and B, one on each of the axes; if U and V are two variable points, one on each axis, so taken that $OU + OV = OA + OB$, find the locus of the intersection of AV and BU.

4. Find the locus of the middle points of the rectangles that may be inscribed in a given triangle.

5. If PP', QQ' are any two parallels to the sides of a given rectangle, find the locus of the intersection of $P'Q$ and PQ'.

CHAPTER III.

THE CIRCLE.

EQUATIONS OF THE CIRCLE.

59. The **Circle** is the locus of a point which moves so that its distance from a fixed point is constant. The fixed point is the **centre,** and the constant distance the **radius,** of the circle.

NOTE. The word "circle," as here defined, means the same thing as "circumference" in Elementary Geometry. This is the usual meaning of "circle" in the higher branches of Mathematics.

60. *To find the equation of a circle, having given its centre (a, b) and its radius r.*

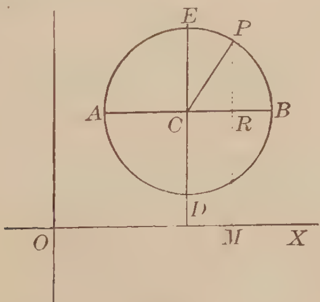


Fig. 24.

Let C (Fig. 24) be the centre, and P any point (x, y) of the circumference. Then it is only necessary to express by an equation the fact that the distance from P to C is constant, and equal to r: the required equation evidently is (§ 6)

$$(x-a)^2+(y-b)^2=r^2. \qquad [15]$$

If we draw $CR \parallel$ to OX, to meet the ordinate of P, then we see from the figure that the legs of the rt. $\triangle\ CPR$ are $CR = x - a$, $PR = y - b$.

If the origin is taken at the centre, then $a = b = 0$, and the equation of the circle is

$$\boldsymbol{x^2 + y^2 = r^2}. \qquad [16]$$

This is the simplest form of the equation of a circle, and the one most commonly used.

If the origin is taken on the circle at the point A, and the diameter AB is taken as the axis of x, then the centre will be the point $(r, 0)$. Writing r in place of a, and 0 in place of b in [15], and reducing, we obtain

$$\boldsymbol{x^2 + y^2 = 2rx}. \qquad [17]$$

Why is this equation without any constant term?

61. *The locus of any equation of the second degree in x and y in which the term in xy is wanting and the coefficients of x^2 and y^2 are equal is a circle.*

Any such equation can evidently be reduced to the form

$$x^2 + y^2 + 2Dx + 2Ey + F = 0. \qquad (1)$$

Therefore,

$$(x^2 + 2Dx + D^2) + (y^2 + 2Ey + E^2) = D^2 + E^2 - F,$$

or $$(x + D)^2 + (y + E)^2 = (D^2 + E^2 - F). \qquad (2)$$

Now, from [15] it follows that the locus of (2) is a circle whose centre is $(-D, -E)$, and whose radius is

$$\sqrt{D^2 + E^2 - F}.$$

Cor. If $D^2 + E^2 > F$, the radius is real and the circle is readily constructed. If $D^2 + E^2 = F$, the radius is zero, and the locus is the single point $(-D, -E)$. If $D^2 + E^2 < F$, the radius is imaginary, and the equation represents no real locus.

62. *Any point* (h, k) *is without, on, or within the circle* $x^2 + y^2 = r^2$, *according as* $h^2 + k^2 >, =,$ or $< r^2$.

For a point is without, on, or within a circle, according as its distance from the centre $>, =,$ or $<$ the radius.

Exercise 21.

Find the equation of the circle, taking as origin :

1. The point B (Fig. 24) ; and BA as axis of x.
2. The point D (Fig. 24) ; and DE as axis of y.
3. The point E (Fig. 24) ; and ED as axis of y.

Write the equations of the following circles:

4. Centre $(5, -3)$, radius 10.
5. Centre $(0, -2)$, radius 11.
6. Centre $(5, 0)$, radius 5.
7. Centre $(-5, 0)$, radius 5.
8. Centre $(2, 3)$, diameter 10.
9. Centre (h, k), radius $\sqrt{h^2 + k^2}$.
10. Determine the centre and radius of the circle

$$x^2 + y^2 - 10x + 12y + 25 = 0.$$

Here $(x - 5)^2 + (y + 6)^2 = 36$. $\therefore a = 5$, $b = -6$, $r = 6$.

Determine the centres and radii of the following circles:

11. $x^2 + y^2 - 2x - 4y = 0$.
12. $3x^2 + 3y^2 - 5x - 7y + 1 = 0$.
13. $x^2 + y^2 - 8x = 0$.
14. $x^2 + y^2 + 8x = 0$.
15. $x^2 + y^2 - 8y = 0$.
16. $x^2 + y^2 + 8y = 0$.
17. $6x^2 - 2y(7 - 3y) = 0$.
18. $x^2 + y^2 = 9k^2$.
19. $(x + y)^2 + (x - y)^2 = 8k^2$.
20. $x^2 + y^2 = a^2 + b^2$.
21. $x^2 + y^2 = k(x + k)$.
22. $x^2 + y^2 = hx + ky$.

23. When are the circles $x^2+y^2+Dx+Ey+C=0$ and $x^2+y^2+D'x+E'y+C'=0$ concentric?

24. What is the geometric meaning of the equation $(x-a)^2+(y-b)^2=0$?

25. Find the intercepts of the circles

$$\text{(i)}\ x^2+y^2-8x-8y+7=0,$$
$$\text{(ii)}\ x^2+y^2-8x-8y+16=0,$$
$$\text{(iii)}\ x^2+y^2-8x-8y+20=0.$$

Putting $y=0$ in each case, we have in case (i) $x^2-8x+7=0$, whence $x=1$ and 7; in case (ii) $x^2-8x+16=0$, whence $x=4$; in case (iii) $x^2-8x+20=0$, whence $x=\pm\sqrt{-4}$.

Putting $x=0$ in each case, we obtain for y values identical with the above values of x.

The geometric meaning of these results is as follows:

Circle (i) cuts the axis of x in the points (1, 0), (7, 0), and the axis of y in the points (0, 1), (0, 7).

Circle (ii) touches the axis of x at (4, 0), and the axis of y at (0, 4).

Circle (iii) does not meet the axes at all.

This is the meaning of the imaginary values of x and y in case (iii).

If, however, we wish to make the language of Geometry conform more exactly to that of Algebra, then in this case we should say that the circle meets the axes in imaginary points. This form of statement, however, must be understood as simply another way of saying that the circle does not meet the axes.

Find the centres, radii, and intercepts on the axes of the following circles:

26. $x^2+y^2-5x-7y+6=0.$

27. $x^2+y^2-12x-4y+15=0.$

28. $x^2+y^2-4x-8y=0.$

29. $x^2+y^2-6x+4y+4=0.$

30. $x^2+y^2+22x-18y+57=0.$

31. Under what conditions will the circle $x^2+y^2+Dx+Ey+C=0$ (i) touch the axis of x? (ii) touch the axis of y? (iii) not meet the axes at all?

32. Show that the circle $x^2+y^2+10x-10y+25=0$ touches the axes and lies entirely in the second quadrant. Write the equation so that it shall represent the same circle touching the axes and lying in the third quadrant.

33. In what points does the straight line $3x+y=25$ cut the circle $x^2+y^2=65$?

34. Find the points common to the loci $x^2+y^2=4$ and $y=2x-4$.

35. The equation of a chord of the circle $x^2+y^2=25$ is $y=2x+11$. Find the length of the chord.

36. The equation of a chord is $\frac{x}{a}+\frac{y}{b}=1$; that of the circle is $x^2+y^2=r^2$. Find the length of the chord.

37. Find the equation of a line passing through the centre of $x^2+y^2-6x-8y=-21$ and perpendicular to $x+2y=4$.

38. Find the equation of that chord of the circle $x^2+y^2=130$ that passes through the point for which the abscissa is 9 and the ordinate negative, and that is parallel to the straight line $4x-5y-7=0$.

39. What is the equation of the chord of the circle $x^2+y^2=277$ that passes through $(3,-5)$ and is bisected at this point?

40. Find the locus of the centre of a circle passing through the points (x_1, y_1) and (x_2, y_2).

41. What is the locus of the centres of all the circles that pass through the points $(5, 3)$ and $(-7, -6)$?

Find the equation of the circle :

42. Passing through the points $(4, 0)$, $(0, 4)$, $(6, 4)$.

43. Passing through the points $(0, 0)$, $(8, 0)$, $(0, -6)$.

44. Passing through the points $(-6, -1)$, $(0, 0)$, $(0, -1)$.

45. Passing through the points $(0, 0)$, $(-8a, 0)$, $(0, 6a)$.

46. Passing through the points $(2, -3)$, $(3, -4)$, $(-2, -1)$.

47. Passing through the points $(1, 2)$, $(1, 3)$, $(2, 5)$.

48. Passing through $(10, 4)$ and $(17, -3)$, and radius $= 13$.

49. Passing through $(3, 6)$, and touching the axes.

50. Touching each axis at the distance 4 from the origin.

51. Touching each axis at the distance a from the origin.

52. Passing through the origin, and cutting the lengths a, b from the axes.

53. Passing through $(5, 6)$, and having its centre at the intersection of the lines $y = 7x - 3$, $4y - 3x = 13$.

54. Passing through $(10, 9)$ and $(5, 2 - 3\sqrt{6})$, and having its centre in the line $3x - 2y - 17 = 0$.

55. Passing through the origin, and cutting equal lengths a from the lines $x = y$, $x + y = 0$.

56. Circumscribing the triangle whose sides are the lines $y = 0$, $y = mx + b$, $\frac{x}{a} + \frac{y}{b} = 1$.

57. Having for diameter the line joining $(0, 0)$ and (x_1, y_1).

58. Having for diameter the line joining (x_1, y_1) and (x_2, y_2).

59. Having for diameter the line joining the points where $y = mx$ meets $x^2 + y^2 = 2rx$.

60. Having for diameter the common chord of the circles $x^2 + y^2 = r^2$ and $(x - a)^2 + y^2 = r^2$.

TANGENTS AND NORMALS.

63. Let QPQ' (Fig. 25) represent any curve. If the secant QPR is turned about the point P until the point Q approaches indefinitely near to P, then *the limiting position*, TT', *of the secant* is called the **Tangent** to the curve at P.

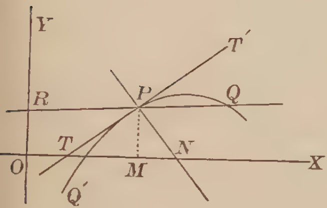


Fig. 25.

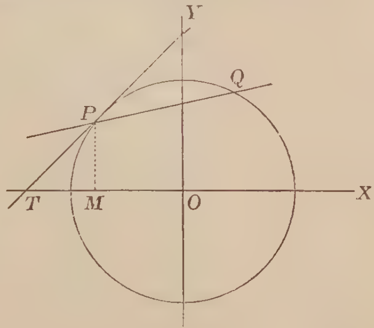


Fig. 26.

The tangent TT' is said to *touch* the curve at P, and the point P is called the **Point of Contact.**

The straight line PN drawn from P, perpendicular to the tangent TT', is called the **Normal** to the curve at P.

Let the curve be referred to the axes OX, OY, and let M be the foot of the ordinate of the point P. Let also the tangent and the normal at P meet the axis of x in the points T, N, respectively. Then TM is called the **Subtangent** for the point P, and MN is called the **Subnormal.**

64. *To find the equation of the tangent to the circle* $x^2+y^2=r^2$, *at the point of contact* (x_1, y_1).

Let P (Fig. 26) be the point (x_1, y_1), and Q any other point (x_2, y_2) of the circle. Then the equation of the secant PQ is

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}. \qquad (1)$$

Now, since (x_1, y_1) and (x_2, y_2) are on this circle, we have

$$x_1^2 + y_1^2 = r^2,$$
$$x_2^2 + y_2^2 = r^2,$$

Subtracting, $\quad (x_2^2 - x_1^2) + (y_2^2 - y_1^2) = 0.$

Factoring, $\quad (x_2 - x_1)(x_2 + x_1) + (y_2 - y_1)(y_2 + y_1) = 0.$

Whence, by transposition and division, we have

$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_2 + x_1}{y_2 + y_1}.$$

By substituting in (1), the equation of the secant becomes

$$\frac{y - y_1}{x - x_1} = -\frac{x_2 + x_1}{y_2 + y_1}.$$

Now let Q coincide with P, or $x_2 = x_1$, $y_2 = y_1$; the secant becomes a tangent at P, and the equation becomes

$$\frac{y - y_1}{x - x_1} = -\frac{x_1}{y_1},$$

or $$x_1 x + y_1 y = x_1^2 + y_1^2.$$

And, since $x_1^2 + y_1^2 = r^2$, we obtain

$$\boldsymbol{x_1 x + y_1 y = r^2}, \qquad [18]$$

which is the equation required.

NOTE. If we had put $x_2 = x_1$, $y_2 = y_1$, in (1) before we introduced the condition that (x_1, y_1) and (x_2, y_2) were on the circle, the slope of the tangent would have assumed the indeterminate form $\frac{0}{0}$.

The above method of obtaining the equation of the tangent to a circle is applicable to any curve whatever. It is sometimes called the *secant* method. Equation [18] is easily remembered from its symmetry, and because it may be formed from $x^2 + y^2 = r^2$ by merely changing x^2 to $x_1 x$, and y^2 to $y_1 y$.

65. *To find the equation of the normal through* (x_1, y_1).

The slope of the tangent is $-\frac{x_1}{y_1}$.

Therefore, that of the normal is $\frac{y_1}{x_1}$ (§ 45, Cor. 2).

Hence, the equation of the normal is (§ 38)

$$y - y_1 = \frac{y_1}{x_1}(x - x_1),$$

which reduces to the form

$$\boldsymbol{y_1 x - x_1 y = 0.} \qquad [19]$$

Therefore, the normal passes through the centre.

66. *To find the equations of the tangent and normal to the circle* $(x-a)^2 + (y-b)^2 = r^2$ *at the point of contact* (x_1, y_1).

We proceed as in § 64, only now the equations of condition which place (x_1, y_1) and (x_2, y_2) on the circle are

$$(x_1 - a)^2 + (y_1 - b)^2 = r^2,$$
$$(x_2 - a)^2 + (y_2 - b)^2 = r^2.$$

After subtracting and factoring, we have

$$(x_2 - x_1)(x_2 + x_1 - 2a) + (y_2 - y_1)(y_2 + y_1 - 2b) = 0,$$

whence, $$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_2 + x_1 - 2a}{y_2 + y_1 - 2b}.$$

Hence, the equation of a secant through (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{x - x_1} = -\frac{x_2 + x_1 - 2a}{y_2 + y_1 - 2b}.$$

Making $x_2 = x_1$, and $y_2 = y_1$, and reducing, we obtain

$$\boldsymbol{(x_1 - a)(x - a) + (y_1 - b)(y - b) = r^2.} \qquad [20]$$

Equation [20] may be immediately formed from [18] by affixing $-a$ to the x factors and $-b$ to the y factors, on the left-hand side.

By proceeding as in § 65, we obtain for the equation of the normal

$$(y_1 - b)(x - x_1) - (x_1 - a)(y - y_1) = 0. \qquad [21]$$

67. *To find the condition that the straight line $y = mx + c$ shall touch the circle $x^2 + y^2 = r^2$.*

I. If the line touches the circle, it is evident that the perpendicular from the origin to the line must be equal to the radius r of the circle. The length of this perpendicular is $\frac{c}{\sqrt{1+m^2}}$ (§ 49). Therefore, the required condition is

$$c^2 = r^2(1 + m^2).$$

II. By eliminating y from the equations

$$y = mx + c, \quad x^2 + y^2 = r^2,$$

we obtain the quadratic in x,

$$(1 + m^2)x^2 + 2mcx = r^2 - c^2,$$

the two roots of which are

$$x = -\frac{mc}{1+m^2} \pm \frac{\sqrt{r^2(1+m^2) - c^2}}{1+m^2}.$$

If these roots are real and unequal, the line will *cut* the circle; if they are equal, it will *touch* the circle; if they are imaginary, it will *not meet the circle at all.*

The roots will be equal if $\sqrt{r^2(1+m^2) - c^2} = 0$; that is, if $c^2 = r^2(1 + m^2)$, a result agreeing with that previously obtained.

If in the equation $y = mx + c$ we substitute for c the value $r\sqrt{1+m^2}$, we obtain the equation of the tangent of a circle in the useful form

$$y = mx \pm r\sqrt{1 + m^2}. \qquad [22]$$

This equation, if we regard m as an arbitrary constant, represents all possible tangents to the circle $x^2 + y^2 = r^2$.

NOTE 1. Method II is applicable to any curve, and agrees with the definition of a tangent given in § 63.

NOTE 2. In problems on tangents the learner should consider whether the coördinates of the point of contact are involved. If they are, he should use equation [18]; if they are not, then in general it is better to use equation [22].

Exercise 22.

1. Explain the meaning of the double sign in equation [22].

2. Deduce the equations of the tangent and normal to the circle $x^2 + y^2 = r^2$, assuming that the normal passes through the centre.

3. Find the equations of the tangent and the normal to $x^2 + y^2 = 52$ at the point (4, 6). Find, also, the lengths of the tangent, normal, subtangent, subnormal, and the portion of the tangent contained between the axes.

4. A straight line touches the circle $x^2 + y^2 = r^2$ in the point (x_1, y_1). Find the lengths of the subtangent, the subnormal, and the portion of the line contained between the axes.

5. What is the equation of the tangent to the circle $x^2 + y^2 = 250$ at the point whose abscissa is 9 and ordinate negative?

6. Find the equations of tangents to $x^2 + y^2 = 10$ at the points whose common abscissa $= 1$.

7. Tangents are drawn through the points of the circle $x^2 + y^2 = 25$ that have abscissas numerically equal to 3. Prove that these tangents enclose a rhombus, and find its area.

8. The subtangent for a certain point of a circle is $5\frac{1}{2}$; the subnormal is 3. What is the equation of the circle?

Find the equation of the straight line:

9. Touching $x^2+y^2=232$ at the point whose abscissa $=14$.

10. Touching $(x-2)^2+(y-3)^2=10$ at the point $(5, 4)$.

11. Touching $x^2+y^2-3x-4y=0$ at the origin.

12. Touching $x^2+y^2-14x-4y-5=0$ at the point whose abscissa is equal to 10.

What is the equation of a straight line touching the circle $x^2+y^2=r^2$, and also:

13. Passing through the point of contact $(r, 0)$?

14. Parallel to the line $Ax+By+C=0$?

15. Perpendicular to the line $Ax+By+C=0$?

16. Making the angle 45° with the axis of x?

17. Passing through the exterior point $(h, 0)$?

18. Forming with the axes a triangle of area r^2?

19. Find the equations of the tangents drawn from the point $(10, 5)$ to the circle $x^2+y^2=100$.

20. Find the equations of tangents to the circle $x^2+y^2+10x-6y-2=0$ and parallel to the line $y=2x-7$.

21. Find the lengths of the subtangent and subnormal in the circle $x^2+y^2-14x-4y=5$ for the point $(10, 9)$.

22. What is the equation of the circle (centre at origin) that is touched by the straight line $x\cos\alpha+y\sin\alpha=p$? What are the coördinates of the point of contact?

23. When will the line $Ax+By-C=0$ touch the circle $x^2+y^2=r^2$? the circle $(x-a)^2+(y-b)^2=r^2$?

24. Find the equation of the straight line touching $x^2+y^2=ax+by$ and passing through the origin.

Prove that the following circles and straight lines touch, and find the point of contact in each case:

25. $x^2+y^2+ax+by=0$ and $ax+by+a^2+b^2=0$.

26. $x^2+y^2-2ax-2by+b^2=0$ and $x=2a$.

27. $x^2+y^2=ax+by$ and $ax-by+b^2=0$.

28. What is the equation of the circle (centre at origin) that touches the line $y=3x-5$?

29. What must be the value of m in order that the line $y=mx+10$ may touch the circle $x^2+y^2=100$? Show that we get the same answer for the line $y=mx-10$, and explain the reason.

30. Determine the value of c in order that the line $3x-4y+c=0$ may touch the circle $x^2+y^2-8x+12y-44=0$. Explain the double answer.

31. What is the equation of the circle having for centre the point (5, 3) and touching the line $3x+2y-10=0$?

32. What is the equation of a circle whose radius $=10$, and which touches the line $4x+3y-70=0$ in the point (10, 10)?

33. A circle touching the line $4x+3y+3=0$ in the point $(-3, 3)$ passes through the point (5, 9). What is its equation?

34. Under what condition will the line $\frac{x}{a}+\frac{y}{b}=1$ touch the circle $x^2+y^2=r^2$?

35. What is the equation of the circle inscribed in the triangle whose sides are

$$x=0, \quad y=0, \quad \frac{x}{a}+\frac{y}{b}=1?$$

36. Two circles touch each other when the distance between their centres is equal to the sum or the difference of their radii. Prove that the circles

$$x^2+y^2=(r+a)^2, \qquad (x-a)^2+y^2=r^2$$

touch each other, and find the equation of the common tangent.

37. Two circles touch each other when the length of their common chord $=0$. Find the length of the common chord of

$$(x-a)^2+(y-b)^2=r^2, \quad (x-b)^2+(y-a)^2=r^2,$$

and hence prove that the two circles touch each other when $(a-b)^2=2r^2$.

Exercise 23. (Review.)

Find the radii and centres of the following circles:

1. $3x^2-6x+3y^2+9y-12=0.$
2. $7x^2+3y^2-4y-(1-2x)^2=0.$
3. $y(y-5)=x(3-x).$
4. $\sqrt{1+a^2}\,(x^2+y^2)=2b(x+ay).$

Find the equation of the circle:

5. Centre $(0, 0)$, radius $=9$.
6. Centre $(7, 0)$, radius $=3$.
7. Centre $(-2, 5)$, radius $=10$.
8. Centre $(3a, 4a)$, radius $=5a$.
9. Centre $(b+c, b-c)$, radius $=c$.
10. Passing through $(a, 0)$, $(0, b)$, $(2a, 2b)$.
11. Passing through $(0, 0)$, $(0, 12)$, $(5, 0)$.
12. Passing through $(10, 9)$, $(4, -5)$, $(0, 5)$.
13. Touching each axis at the distance -7 from the origin.

14. Touching both axes, and radius $= r$.

15. Centre (a, a), and cutting chord $= b$ from each axis.

16. Having the centre $(0, 0)$, and touching $y = 2x + 3$.

17. Having the centre $(1, -3)$, and touching $2x - y = 4$.

18. With its centre in the line $5x - 7y - 8 = 0$, and touching the lines $2x - y = 0$, $x - 2y - 6 = 0$.

19. Passing through the origin and the points common to the circles

$$x^2 + y^2 - 6x - 10y - 15 = 0,$$
$$x^2 + y^2 + 2x + 4y - 20 = 0.$$

20. Having its centre in the line $5x - 3y - 7 = 0$, and passing through the points common to the same circles as in No. 19.

21. Touching the axis of x, and passing through the points common to the circles

$$x^2 + y^2 + 4x - 14y - 68 = 0,$$
$$x^2 + y^2 - 6x - 22y + 30 = 0.$$

22. Find the centre and the radius of the circle which passes through $(9, 6)$, $(10, 5)$, $(3, -2)$.

23. What is the distance from the centre of the circle passing through $(2, 0)$, $(8, 0)$, $(5, 9)$ to the straight line joining $(0, -11)$ and $(-16, 1)$?

24. What is the distance from the centre of the circle $x^2 + y^2 - 4x + 8y = 0$ to the line $4x - 3y + 30 = 0$?

25. What portion of the line $y = 5x + 2$ is contained within the circle $x^2 + y^2 - 13x - 4y - 9 = 0$?

26. Through that point of the circle $x^2 + y^2 = 25$ for which the abscissa $= 4$ and the ordinate is negative, a straight line parallel to $y = 3x - 5$ is drawn. Find the length of the intercepted chord.

27. Through the point (x_1, y_1), within the circle $x^2+y^2=r^2$, a chord is drawn so as to be bisected at this point. What is its equation?

28. What relation must exist among the coefficients of the equation $A(x^2+y^2)+Dx+Ey+C=0$,

(i) in order that the circle may touch the axis of x?
(ii) in order that the circle may touch the axis of y?
(iii) in order that the circle may touch both axes?

29. Under what condition will the straight line $y=mx+c$ touch the circle $x^2+y^2=2rx$?

30. What must be the value of k in order that the line $3x+4y=k$ may touch the circle $y^2=10x-x^2$?

31. Find the equation of the circle that passes through the origin and cuts equal lengths a from the lines $x=y$, $x+y=0$.

32. Find the equations of the four circles whose common radius $=a\sqrt{2}$, and which cut chords from each axis equal to $2a$.

33. Find the equation of the circle whose diameter is the common chord of the circles $x^2+y^2=r^2$, $(x-a)^2+y^2=r^2$.

Find the equation of the straight line:

34. Passing through $(0, 0)$ and the centre of the circle
$$x^2+y^2=a(x+y).$$

35. Passing through the centres of the circles
$$x^2+y^2=25 \text{ and } x^2+y^2+6x-8y=0.$$

36. Passing through $(0, 0)$ and touching the circle
$$x^2+y^2-6x-12y+41=0.$$

37. Parallel to $x+\sqrt{3}(y-12)=0$ and touching $x^2+y^2=100$.

38. Passing through the points common to the circles

$$x^2+y^2-2x-4y-20=0,$$
$$x^2+y^2-14x-16y+100=0.$$

39. Prove that the common chord of the circles in No. 38 is perpendicular to the straight line joining their centres.

40. Find the area of the triangle formed by the radii of the circle $x^2+y^2=169$ drawn to the points whose abscissas are -12 and $+7$ and ordinates positive, and the chord passing through the same two points.

41. Prove that an angle inscribed in a semicircle is a right angle.

42. Prove that the radius of a circle drawn perpendicular to a chord bisects the chord.

43. Find the inclination to the axis of x of the line joining the centres of the circles $x^2+2x+y^2=0$, $x^2+2y+y^2=0$.

44. Determine the point from which tangents drawn to the circles

$$x^2+y^2-2x-6y+6=0,$$
$$x^2+y^2-22y-20x+52=0,$$

will each be equal to $4\sqrt{6}$.

45. Find the equations of the circles that touch the straight lines $6x+7y+9=0$ and $7x+6y+3=0$, the latter line in the point $(3,-4)$.

Obtain and discuss the equations of the following loci:

46. Locus of the centre of a circle having the radius r and passing through the point (x_1, y_1).

47. Locus of the centre of a circle having the radius r' and touching the circle $(x-a)^2+(y-b)^2=r^2$.

48. Locus of all points from which tangents drawn to the circle $(x-a)^2+(y-b)^2=r^2$ have a given length t.

49. Locus of the middle point of a chord drawn through a fixed point A of a given circle.

50. Locus of the point M which divides the chord AC, drawn through the fixed point A of a given circle, in a given ratio $AM : MC = m : n$.

51. Locus of a point whose distances from two fixed points, A, B, are in a constant ratio $m : n$.

52. Locus of a point, the sum of the squares of whose distances from two fixed points, A, B, is constant, and equal to k^2.

53. Locus of a point, the difference of the squares of whose distances from two fixed points, A, B, is constant and equal to k^2.

54. Locus of the middle point of a line of constant length d which moves so that its ends always touch two fixed perpendicular lines.

55. Locus of the vertex of a triangle whose base is fixed and of constant length, and the angle at the vertex is also constant.

56. One side, AB, of a triangle is constant in length and fixed in position; another side, AC, is constant in length but revolves about the point A. Find the locus of the middle point of the third side, BC.

57. Find the locus of the intersections of tangents at the extremities of a chord whose length is constant.

58. A and B are two fixed points, and the point P moves so that $PA = n \times PB$; find the locus of P.

SUPPLEMENTARY PROPOSITIONS.

68. A **Diameter** of a curve is the locus of the middle points of a system of parallel chords. The chords which any diameter bisects are called the **Chords** of that diameter.

69. *To find the equation of a diameter of the circle* $x^2 + y^2 = r^2$.

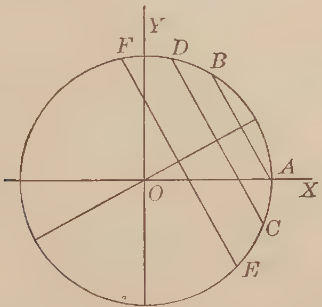


Fig. 27.

Let the equation of any one of the parallel chords (Fig. 27) be $y = mx + c$, and let it meet the circle in the points (x_1, y_1) and (x_2, y_2).

Then (§§ 37 and 64) $m = -\dfrac{x_1 + x_2}{y_1 + y_2}$. (1)

Let (x, y) be the middle point of the chord; then $2x = x_1 + x_2$, $2y = y_1 + y_2$ (§ 8), and by substitution we have

$$m = -\frac{x}{y},$$

or

$$y = -\frac{1}{m}x, \qquad (2)$$

a relation which evidently holds true for the middle points of all the chords. Therefore (2) is the equation of a diameter.

Cor. From (2) we see that a diameter of a circle is a straight line passing through the centre and perpendicular to its chords.

70. *Two distinct, two coincident, or no tangents can be drawn to a circle through any point (h, k), according as this point is without, on, or within the circle.*

Let the tangent

$$y = mx + r\sqrt{1 + m^2}$$

pass through the point (h, k); then

$$k = mh + r\sqrt{1 + m^2}.$$

Transposing and squaring, we have

$$(h^2 - r^2)m^2 - 2hkm = r^2 - k^2.$$

$$\therefore m = \frac{hk \pm r\sqrt{h^2 + k^2 - r^2}}{h^2 - r^2}. \tag{1}$$

The values of m given in (1) are the slopes of the tangents through (h, k). Now, these values are real and unequal, real and equal, or imaginary, according as $h^2 + k^2 >, =,$ or $< r^2$; that is, according as (h, k) is without, on, or within the circle. Hence, two distinct, two coincident, or no tangents can be drawn through (h, k), according as (h, k) is without, on, or within the circle.

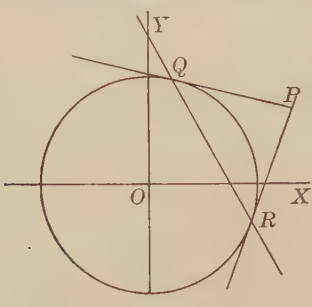


Fig. 28.

71. *To find the equation of the chord joining the points of contact of the two tangents from any external point (h, k).*

Let (x_1, y_1), (x_2, y_2) be the points of contact Q and R; then the equations of the tangents PQ and PR are (§ 64)

$$x_1x + y_1y = r^2,$$
$$x_2x + y_2y = r^2.$$

Since both tangents pass through $P(h, k)$, both these equations are satisfied by the coördinates h, k; therefore,

$$hx_1 + ky_1 = r^2, \tag{1}$$
$$hx_2 + ky_2 = r^2. \tag{2}$$

From equations (1) and (2) we see that the coördinates of both the points (x_1, y_1), (x_2, y_2) satisfy the equation

$$hx + ky = r^2. \tag{3}$$

Hence, the locus of (3), which is a right line, passes through both points of contact; and, therefore, (3) is the equation of the chord QR. The chord QR is called the **Chord of Contact.**

72. *Suppose a chord of a circle to turn round any fixed point* (h, k) *; to find the locus of the intersection of the two tangents drawn at its extremities.*

Let P (Fig. 29 or 30) be the fixed point (h, k), QPR one position of the revolving chord, and let the tangents at Q and R intersect in P_1, (x_1, y_1); it is required to find the locus of P_1 as the chord turns about P. Since QR is the chord of contact of tangents drawn from the point P_1 (x_1, y_1), its equation is (§ 71)

$$x_1x + y_1y = r^2. \tag{1}$$

Since (1) passes through (h, k), we have

$$hx_1 + ky_1 = r^2. \tag{2}$$

But (x_1, y_1) is *any* point in the required locus, and by (2) its coördinates satisfy the equation

$$hx + ky = r^2; \tag{3}$$

hence, (3) is the equation of the required locus.

Since (3) is of the first degree, the locus is a straight line.

The line $hx + ky = r^2$ is called the **Polar** of the point (h, k) with regard to the circle $x^2 + y^2 = r^2$, and the point (h, k) is called the **Pole** of the line. The pole (h, k) may be without, on, or within the curve. In Fig. 29 it is within, while in Fig. 30 it is without the circle.

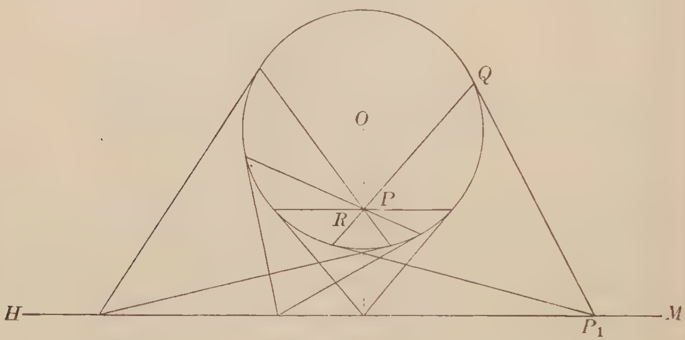


Fig. 29.

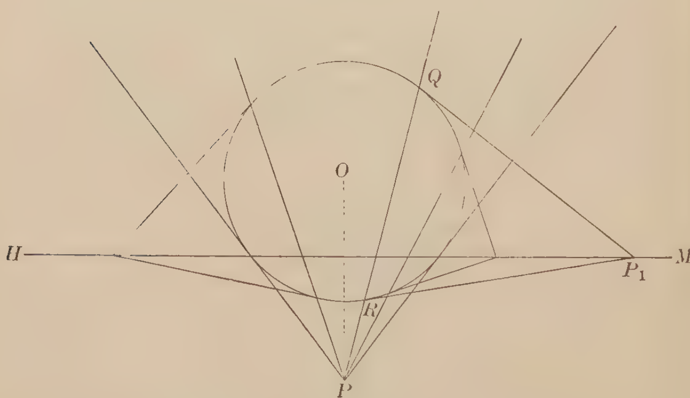


Fig. 30.

COR. 1. If the point (h, k) is on the circle, (3) is evidently the equation of the tangent at (h, k); hence,

The polar of any point on the circle is identical with the tangent at that point.

COR. 2. If (h, k) is an external point, by § 71, (3) is the equation of the chord of contact of tangents from (h, k) to the circle; hence,

The polar of any external point is the same line as the chord of contact of tangents drawn from that point.

Thus, in Fig. 30, HM is the polar of P, or the chord of contact of tangents drawn from P.

73. The polar and pole of a circle may be defined as follows: If a chord of a circle is turned round a fixed point (h, k), the locus of the intersection of the two tangents at its extremities is the polar of the pole (h, k) with regard to that curve.

74. *If the polar of a point P passes through P', then the polar of P' will pass through P.*

Let P be the point (h, k), P' the point (h', k'), and let the equation of the circle be $x^2 + y^2 = r^2$.

Then the equations of the polars of P and P' are

$$hx + ky = r^2, \tag{1}$$

$$h'x + k'y = r^2. \tag{2}$$

If P' is on the polar of P, its coördinates must satisfy equation (1); therefore,

$$hh' + kk' = r^2.$$

But this is also the condition that P shall be on the line represented by (2); that is, on the polar of P'. Therefore, P is on the polar of P'.

This relation of poles and polars is illustrated in the Figs. 29 and 30.

75. *To find a geometrical construction for the polar of a point with respect to a circle.*

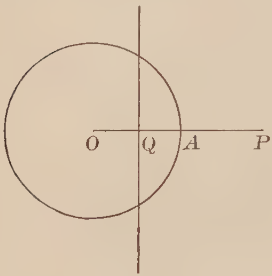


Fig. 31.

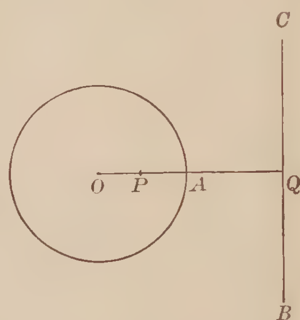


Fig. 32.

The equation of the line through any point P (h, k) and the centre of the circle, or the origin, is

$$kx - hy = 0. \tag{1}$$

Now, the equation of the polar of P is

$$hx + ky = r^2. \tag{2}$$

But the loci of (1) and (2) are perpendicular (§ 45, Cor. 2).

Hence, if BC is the polar of P, OP is perpendicular to BC. and

$$OQ = \frac{r^2}{\sqrt{h^2 + k^2}}. \qquad (\S\ 41)$$

Also, $\qquad OP = \sqrt{h^2 + k^2}.$

Therefore, $\qquad OP \times OQ = r^2.$

Hence, to construct the polar of P:

Join OP, and let it cut the circle in A; take Q in the line OP, so that

$$OP : OA = OA : OQ.$$

The line through Q perpendicular to OP is the polar of P.

To locate the pole of BC, draw OQ perpendicular to BC, and take P so that

$$OQ : OA = OA : OP.$$

76. *To find the length of the tangent drawn from any point (h, k) to the circle* $(x-a)^2+(y-b)^2-r^2=0$. (1)

Let P (Fig. 33) be the point (h, k), Q the point of contact, C the centre of the circle; then, since PQC is a right angle,

$$\overline{PQ}^2=\overline{PC}^2-\overline{QC}^2.$$

Now, $\overline{PC}^2=(h-a)^2+(k-b)^2$, and $\overline{QC}^2=r^2$.

Therefore, $\overline{PQ}^2=(h-a)^2+(k-b)^2-r^2$.

Hence $\overline{PQ}^2$ is found by simply substituting the coördinates of P for x and y in the expression $(x-a)^2+(y-b)^2-r^2$.

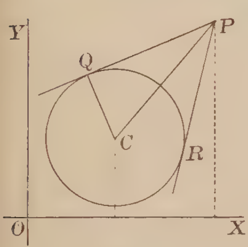


Fig. 33.

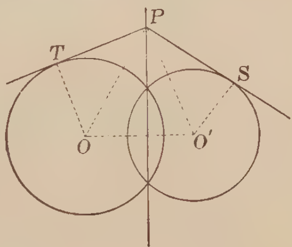


Fig. 34.

If for brevity we write S instead of $(x-a)^2+(y-b)^2-r^2$, then the equation $S=0$ will represent the general equation of the circle after division by the common coefficient of x^2 and y^2, and we may state the above result as follows:

If $S=0$ is the equation of a circle, and the coördinates of any point are substituted for x and y in S, the result will be equal to the square of the length of the tangent drawn from the point to the circle.

77. *To find the locus of the point from which tangents drawn to two given circles are equal.*

Let the equations of the circles O and O' (Fig. 34), be

$$(x-a)^2+(y-b)^2-r^2=0, \qquad (1)$$

and

$$(x-a')^2+(y-b')^2-r'^2=0. \qquad (2)$$

Then, if the tangents drawn from $P(x, y)$ to the circles (1) and (2) are equal, we have

$$(x-a)^2+(y-b)^2-r^2=(x-a')^2+(y-b')^2-r'^2, \qquad (3)$$

which is the equation of the required locus.

Cor. 1. Performing the indicated operations in (3), and transposing, we have

$$2(a-a')x+2(b-b')y=a^2-a'^2+b^2-b'^2-r^2+r'^2, \quad (4)$$

which shows that the locus is a straight line.

This locus is called the **Radical Axis** of the two circles.

Hence, *if* $S_1=0$, $S_2=0$ *are the equations of two circles, then*

$$S_1=S_2, \text{ or } S_1-S_2=0,$$

will be the equation of their radical axis.

Cor. 2. When the circles $S_1=0$ and $S_2=0$ intersect, the locus of $S_1=S_2$ passes through their common points.

Hence, *when two circles intersect or are tangent, their radical axis is their common chord or common tangent.*

Cor. 3. The slope of (4) is the negative reciprocal of the slope of the line joining the centres of (1) and (2).

Hence, *the radical axis of two circles is perpendicular to the line joining their centres.*

78. Let $S=0$, $S_1=0$, $S_2=0$ be the equations of three circles, in each of which the coefficient of x^2 is unity.

Then the equations of their radical axes, taken in pairs, are

$$S-S_1=0, \quad S_1-S_2=0, \quad S-S_2=0.$$

The values of x and y that will satisfy any two of these equations will also satisfy the third. Therefore, the third axis passes through the point common to the other two. Hence,

The three radical axes of three circles, taken in pairs, meet in a point. This point is called the **Radical Centre** of the three circles.

Exercise 24.

1. What is the equation of the diameter of the circle $x^2+y^2=20$ that bisects chords parallel to the line $6x+7y+8=0$?

2. What is the equation of the diameter of the circle that bisects all chords whose inclination to the axis of x is $135°$?

3. Prove that the tangents at the extremities of a diameter are parallel.

4. Write the equations of the chords of contact in the circle $x^2+y^2=r^2$ for tangents drawn from the following points: (r, r), $(2r, 3r)$, $(a+b, a-b)$.

5. From the point $(13, 2)$ tangents are drawn to the circle $x^2+y^2=49$; what is the equation of the chord of contact?

6. What line is represented by the equation $hx+ky=r^2$ when (h, k) is on the circle?

7. Write the equations of the polars of the following points with respect to the circle $x^2+y^2=4$:

(i) $(2, 3)$. (ii) $(3, -1)$. (iii) $(1, -1)$.

8. Find the poles of the following lines with respect to the circle $x^2+y^2=35$:

(i) $4x+6y=7$. (ii) $3x-2y=5$. (iii) $ax+by=1$.

9. Find the pole of $3x+4y=7$ with respect to the circle $x^2+y^2=14$.

10. Find the pole of $Ax+By+C=0$ with respect to the circle $x^2+y^2=r^2$.

11. Find the coördinates of the points in which the line $x=4$ cuts the circle $x^2+y^2=25$; also find the equations of the tangents at those points, and show that they intersect in the point $(\frac{25}{4}, 0)$.

12. If the polars of two points P, Q meet in R, then R is the pole of the line PQ.

13. If the polar of (h, k) with respect to the circle $x^2+y^2=r^2$ touches the circle $x^2+y^2=2rx$, then $k^2+2rh=r^2$.

14. If the polar of (h, k) with respect to the circle $x^2+y^2=c^2$ touches the circle $4(x^2+y^2)=c^2$, then the pole (h, k) will lie on the circle $x^2+y^2=4c^2$.

15. Find the polar of the centre of the circle $x^2+y^2=r^2$. Trace the changes in the position of the polar as the pole is supposed to move from the centre to an infinite distance.

16. What is the square of the length of the tangent drawn from the point (h, k) to the circle $x^2+y^2=r^2$?

17. Find the length of the tangent drawn from $(2, 5)$ to the circle $x^2+y^2-2x-3y-1=0$.

Find the radical axis of the circles:

18. $(x+5)^2+(y+6)^2=9$, $(x-7)^2+(y-11)^2=16$.

19. $x^2+y^2+2x+3y-7=0$, $x^2+y^2-2x-y+1=0$.

20. $x^2+y^2+bx+by-c=0$, $ax^2+ay^2+a^2x+b^2y=0$.

21. Find the radical axis and length of the common chord of the circles

$$x^2+y^2+ax+by+c=0,\ x^2+y^2+bx+ay+c=0.$$

22. Find the radical centre of the three circles

$$x^2+y^2+4x+7=0,$$
$$2x^2+2y^2+3x+5y+9=0,$$
$$x^2+y^2+y=0.$$

CHAPTER IV.

DIFFERENT SYSTEMS OF COÖRDINATES.

Rectilinear System.

79. When we define the position of a point, with reference to any fixed lines or points, we are said to use a **System of Coördinates.**

In the **Rectilinear System,** already described, we have thus far employed only rectangular axes, or coördinates, which are to be preferred for most purposes, on account of their greater simplicity. When the axes of reference intersect at oblique angles, the axes and coördinates are called **Oblique.**

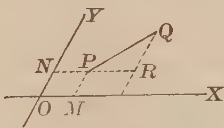


Fig. 35.

Let OX, OY (Fig. 35) be two axes making an acute angle, $XOY = \omega$, with each other. If we draw $PN \parallel$ to OX, and $PM \parallel$ to OY, then the coördinates of P are

$$NP = OM = x, \quad MP = y.$$

Since oblique and rectangular coördinates differ only in the angle included between the axes, any of the previously deduced formulas that do not depend on any property of the right angle are applicable when the axes are oblique. Thus, formulas [2], [3], [4], [7] hold for oblique axes as

well as for rectangular, and therefore are general formulas for the Rectilinear System.

When the axes are oblique, instead of [1], we evidently have (Fig. 35)

$$PQ = \sqrt{\overline{PR}^2 + \overline{RQ}^2 - 2PR \times RQ \cos PRQ},$$

$$\therefore d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1) \cos \omega},$$

which reduces to [1] when $\omega = 90°$.

The Rectilinear System is sometimes called the **Cartesian System**, from Descartes, who first used it.

80. *To find the equation of the straight line AC, referred to the oblique axes OX, OY* (Fig. 36), *having given the intercept $OB = b$ and the angle $XAC = \gamma$.*

Let P be any point (x, y) of the line. Draw $BD \parallel$ to OX, meeting PM in D. Then, by Trigonometry,

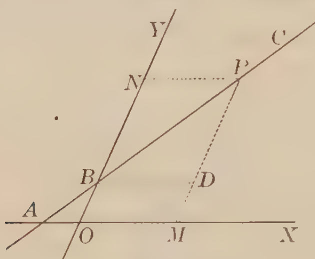


Fig. 36.

$$\frac{PD}{BD} = \frac{\sin \gamma}{\sin (\omega - \gamma)}, \text{ or } \frac{y - b}{x} = \frac{\sin \gamma}{\sin (\omega - \gamma)}.$$

If now we put $m = \dfrac{\sin \gamma}{\sin (\omega - \gamma)}$, we obtain as the result an equation of the same form as [6], p. 38,

$$y = mx + b.$$

Here $m =$ the ratio of the sines of the angles which the line AC makes with the axes; that is, $m = \sin XAP \div \sin PBY$, which equals $\tan XAP = \tan \gamma$ when $\omega = 90°$.

81. Oblique coördinates are seldom used, because they generally lead to more complex formulas than rectangular coördinates. In many cases, however, they may be employed to advantage. An example of this kind is furnished by problem No. 23, p. 65:

To prove that the medians of a triangle meet in one point.

If a, b, c represent the three sides of the triangle, and we take as axes the sides a and b, then the equations of the sides and also of the medians may be written with great ease, as follows:

The sides, $y=0, \quad x=0, \quad \frac{x}{a}+\frac{y}{b}=1.$

The medians,

$$\frac{2x}{a}+\frac{y}{b}-1=0, \quad \frac{x}{a}+\frac{2y}{b}-1=0, \quad \frac{x}{a}-\frac{y}{b}=0.$$

On comparing the equations of the medians, we see that if we subtract the second equation from the first, we obtain the third; therefore, the three medians must pass through the same point (§ 53).

Polar System of Coördinates.

82. Next to the rectilinear, the system of coördinates most frequently used is the **Polar System.**

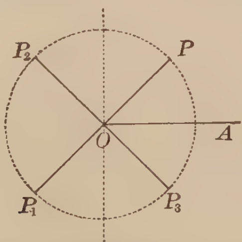


Fig. 37.

Let O (Fig. 37) be a fixed point, OA a fixed straight line, P any point. Join OP.

It is evident that we know the position of P, provided we know the distance OP and the angle which OP forms with OA.

Thus, if we denote the distance OP by ρ, and the angle AOP by θ, the position of P is determined if ρ and θ are known.

The fixed point O is called the **Pole,** and the fixed line OA the **Polar Axis;** ρ and θ are called the **Polar Coördinates** of P; ρ, its **Radius Vector;** and θ, its **Direction** or **Vectorial Angle.**

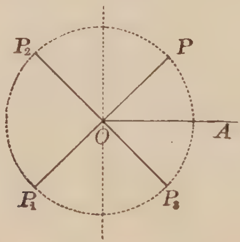


Fig. 38.

Every point in a plane is perfectly determined by a positive value of ρ between 0 and ∞, and a positive value of θ between 0° and 360° (or 0 and 2π, circular measure). But, in order to represent by a single equation all the points of a geometric locus, we often employ negative values of ρ and θ, and adopt the following laws of signs:

(i) θ is positive when measured from right to left, and negative when measured in the opposite direction.

(ii) ρ is positive or negative according as it extends in the direction of the terminal side of θ or in the opposite direction. Thus, any given point may be determined in four different ways.

For example, suppose that the straight line POP_1 bisects the first and third quadrants, and that in this line we take points P, P_1, at the same distance $OP = a$ from O; then

P is the point $(a, \frac{1}{4}\pi)$ or $(-a, \frac{5}{4}\pi)$ or $(-a, -\frac{3}{4}\pi)$ or $(a, -\frac{7}{4}\pi)$;
P_1 is the point $(a, \frac{5}{4}\pi)$ or $(-a, \frac{1}{4}\pi)$ or $(a, -\frac{3}{4}\pi)$ or $(-a, -\frac{7}{4}\pi)$.

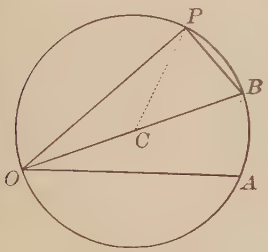


Fig. 39.

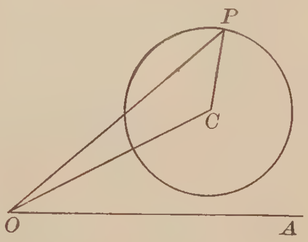


Fig. 40.

83. *To find the polar equation of a circle.*

(i) Let the pole O be at the centre (Fig. 38). Then, if r denotes the radius, the polar equation is simply $\rho = r$.

(ii) Let the pole O be on the circumference (Fig. 39), and let the diameter OB make an angle α with the initial line OA. Let P be any point (ρ, θ) of the circle. Join BP.

Then, $$OP = OB \cos BOP,$$

or $$\boldsymbol{\rho = 2r \cos(\theta - \alpha).} \qquad [23]$$

If OB is taken as the initial line, the equation becomes

$$\boldsymbol{\rho = 2r \cos\theta.} \qquad [24]$$

(iii) Let the pole O be any point, and the centre the point (ρ', θ'). Then in the triangle OCP (Fig. 40),

$$\overline{OP}^2 - 2OP \times OC \times \cos COP + \overline{OC}^2 - \overline{CP}^2 = 0,$$

or $$\boldsymbol{\rho^2 - 2\rho\rho' \cos(\theta - \theta') + \rho'^2 - r^2 = 0,} \qquad [25]$$

the most general form of the polar equation of a circle.

Exercise 25.

1. Find the distances from the point P in Fig. 38 to the two axes.

2. Prove that the equation of a straight line, referred to oblique axes in terms of its intercepts, is identical in form with [7], p. 39.

3. If the straight line P_2OP_3 (Fig. 38) bisects the second and fourth quadrants, what are the polar coördinates of the points P_2 and P_3? Give more than one set of values in each case.

4. Construct the following points (on paper, take $a = 1$ in.):

$$\left(a, 0\right), \left(a, \frac{\pi}{2}\right), \left(a, -\frac{\pi}{2}\right), \left(-a, \frac{\pi}{2}\right), \left(-a, -\frac{\pi}{2}\right),$$
$$\left(2a, \frac{\pi}{6}\right), \left(2a, \pi\right), \left(a \cos\frac{\pi}{3}, \frac{\pi}{3}\right), \left(a, \frac{3\pi}{2}\right), \left(3a, \frac{2\pi}{3}\right),$$
$$\left(-3a, \frac{2\pi}{3}\right), \left(4a, \tan^{-1}\frac{4}{3}\right), \left(4a, \tan^{-1}\frac{3}{4}\right).$$

NOTE. The expression $\tan^{-1}\frac{4}{3}$ in higher Mathematics means *the angle whose tangent is* $\frac{4}{3}$.

5. If ρ_1, ρ_2 denote the two values of ρ in equation [25], p. 103, prove that $\rho_1\rho_2 = \rho'^2 - r^2$. What theorem of Elementary Geometry is expressed by this equation if the pole is outside the circle? if the pole is inside the circle?

6. Through a fixed point P in a circle a chord PB is drawn, and then revolved about P; find the locus of its middle point.

NOTE. In such problems as this there is a great advantage in using polar equations.

7. If p denotes the distance from the pole to a straight line, α the angle between p and the polar axis, prove that the polar equation of the line is $\rho \cos(\theta - \alpha) = p$.

TRANSFORMATION OF COÖRDINATES.

84. The equation of a curve is oftentimes greatly simplified by referring it to a new set of axes, or to a new system of coördinates. For example, compare equations [15] and [16], p. 71. Hence, it is sometimes useful to be able to deduce from the equation of a curve referred to one set of axes or to one system of coördinates, its equation when referred to another set of axes or to another system of coördinates. Either of these operations is known as the **Transformation of Coördinates.** It consists of expressing the old coördinates in terms of the new, and then replacing in the equation of the curve the old coördinates by their values in terms of the new; we thus obtain a constant relation between the new coördinates, that will represent the curve referred to the new axes or system.

85. *To change the origin to the point* (h, k) *without changing the direction of the axes.*

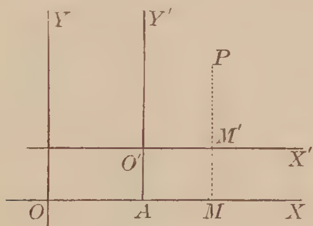


Fig. 41.

Let OX, OY be the old axes, $O'X'$, $O'Y'$ the new; and let (x, y), (x', y') be the coördinates of the *same* point P, referred to the old and new axes respectively.

Then (Fig. 41)

$$OA = h,\ AO' = k,\ OM = x,\ MP = y,\ O'M' = x',\ M'P = y'.$$

$$x = OA + AM = OA + O'M' = x' + h.$$

$$y = MM' + M'P = AO' + M'P = y' + k.$$

These relations are equally true for rectangular and oblique coördinates.

Hence, to find what the equation of a curve becomes when the origin is transferred to a point (h, k), the new axes running parallel to the old, we must substitute for x and y the values given above.

After the substitution, we may write x and y instead of x' and y'; so that practically the change is effected *by simply writing* $x + h$ *in place of* x, $y + k$ *in place of* y.

If, however, we wish to transform a *point* (x, y) from the new to the old system, we must write $x - h$ in place of x and $y - k$ in place of y.

86. *To change the reference of a curve from one set of rectangular axes to another, the origin remaining the same.*

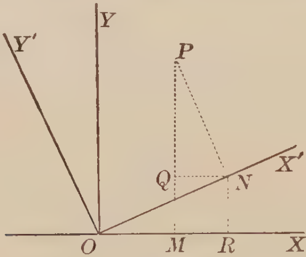


Fig. 42.

Let (x, y) be a point P referred to the old axes OX, OY; (x', y') the same point referred to the new axes OX', OY' (Fig. 42). Then

$$OM = x,\ MP = y,\ ON = x',\ NP = y'.$$

Let the angle $XOX' = \theta$. Draw NQ, $NR \perp$ to PM, OX, respectively; then

$$NPQ = QNO = RON = \theta.$$

Hence, $OM = OR - RM = OR - NQ = ON \cos\theta - PN \sin\theta$.

Or $\qquad x = x' \cos\theta - y' \sin\theta$.

And $\quad PM = MQ + QP = RN + QP = ON \sin\theta + PN \cos\theta$.

Or $\qquad y = x' \sin\theta + y' \cos\theta$.

Therefore, to find what the equation of a curve becomes when referred to the new axes, we must write

$$x \cos\theta - y \sin\theta \text{ for } x, \quad x \sin\theta + y \cos\theta \text{ for } y.$$

87. *To change the reference of a curve from one set of rectangular axes to another, both the origin and the direction of the axes being changed.*

First transform the equation to axes through the new origin, parallel to the old axes. Then turn these axes through the required angle.

If (h, k) is the new origin referred to the old axes, θ the angle between the old and new axes of x, we obtain as the values of x and y for any point P, in terms of the new coördinates,

$$x = h + x' \cos\theta - y' \sin\theta,$$
$$y = k + x' \sin\theta + y' \cos\theta,$$

In making all these transformations, attention must be paid to the *signs* of h, k, and θ.

88. *To change the reference of a curve from rectangular to oblique axes, the origin remaining the same.*

Let α, β be the angles formed by the *positive* directions of the new axes OX', OY' (Fig. 43) with the positive direction of OX. Let the old coördinates of a point P be x, y; and the new coördinates, x', y'. Then from the right triangles ORN, PQN we readily obtain the formulas

$$x = x' \cos\alpha + y' \cos\beta,$$
$$y = x' \sin\alpha + y' \sin\beta.$$

Investigate the special case when $\beta = \alpha + 90°$.

89. *To deduce the formulas for finding the polar equation of a curve from its rectangular equation.*

Let x, y be the rectangular coördinates of any point P, and ρ, θ its polar coördinates.

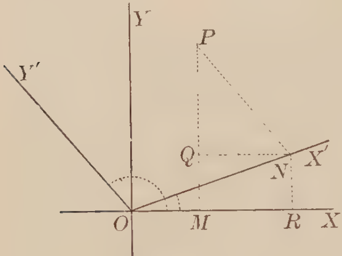


Fig. 43.

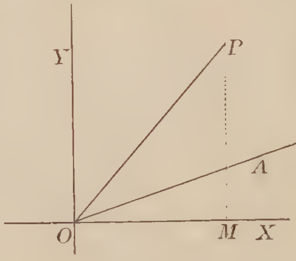


Fig. 44.

(i) Let the origin of rectangular coördinates be the pole, and let the polar axis coincide with the axis of x.

Then (Fig. 44)

$$OM = OP \cos MOP,$$
$$PM = OP \sin MOP.$$

Or

$$x = \rho \cos \theta,$$
$$y = \rho \sin \theta.$$

(ii) If the pole is the point (h, k), we have

$$x = h + \rho \cos \theta,$$
$$y = k + \rho \sin \theta.$$

(iii) If the pole coincides with the origin, but the polar axis OA makes the angle α with the axis of x, we obtain

$$x = \rho \cos (\theta + \alpha),$$
$$y = \rho \sin (\theta + \alpha).$$

(iv) If the pole is the point (h, k), and the polar axis makes the angle α with the axis of x,

$$x = h + \rho \cos (\theta + \alpha);$$
$$y = k + \rho \sin (\theta + \alpha).$$

90. *To deduce the formulas for finding the rectangular equation of a curve from its polar equation.*

From the results in cases (i) and (ii) of § 89 (the only cases of importance), we readily obtain

In case (i), $\rho^2 = x^2 + y^2$, $\quad \tan\theta = \dfrac{y}{x}.$

In case (ii), $\rho^2 = (x-h)^2 + (y-k)^2$, $\tan\theta = \dfrac{y-k}{x-h}.$

91. *The degree of an equation is not altered by passing from one set of axes to another.*

For, however the axes may be changed, the new equation is always obtained by substituting for x and y expressions of the form

$$ax + by + c \text{ and } a'x + b'y + c'.$$

These expressions are of the first degree, and, therefore, if they replace x and y in the equation, the degree of the equation cannot be *raised*. Neither can it be *lowered*; for if it could be lowered, it would be raised by returning to the original axes, and therefore to the original equation.

Exercise 26.

1. What does the equation $y^2 - 4x + 4y + 8 = 0$ become when the origin is changed to the point $(1, -2)$?

Transform the equation of the circle $(x-a)^2 + (y-b)^2 = r^2$ by changing the origin:

2. To the centre of the circle.

3. To the left-hand end of the horizontal diameter.

4. To the upper end of the vertical diameter.

5. What does the equation $x^2 + y^2 = r^2$ become if the axes are turned through the angle α?

6. What does the equation $x^2 - y^2 = a^2$ become if the axes are turned through $-45°$?

7. The equation of a curve referred to rectangular axes is $x-xy-y=0$. Transform it to new axes, whose origin is the point $(-1, 1)$, the new axis of y bisecting two of the angles formed by the old axes.

8. Change the following equations to polar coördinates, taking the pole at the origin and the polar axis to coincide with the axis of x:

(i) $x^2+y^2=a^2$. (ii) $x^2-y^2=a^2$.

9. Change the equation $x^2=4ay$ to polar coördinates, (i) taking the pole at the origin; (ii) taking the pole at the point $(a, 0)$.

10. Change the following equations to rectangular coördinates, the origin coinciding with the pole, and the polar axis with the axis of x:

(i) $\rho=a$, (ii) $\rho=a\cos\theta$, (iii) $\rho^2\cos 2\theta=a^2$.

Transform the following equations by changing the origin to the point given as a new origin:

11. $x+y+2=0$; the new origin $(-2, 0)$.

12. $2x-5y-10=0$; the new origin $(5, -2)$.

13. $3x^2+4xy+y^2-5x-6y-3=0$; new origin $(\frac{7}{2}, -4)$.

14. $x^2+y^2-2x-4y=20$; new origin $(1, 2)$.

15. $x^2-6xy+y^2-6x+2y+1=0$; new origin $(0, -1)$.

16. Transform the equation $x^2-y^2+6=0$ by turning the axes through 45°.

17. Transform the equation $(x+y-2a)^2=4xy$ by turning the axes through 45°.

18. Transform the equation $9x^2-16y^2=144$ to oblique axes, such that the new axis of x makes with the old axis of x the negative angle $\tan^{-1}-\frac{3}{4}$; and the new axis of y makes with the old axis of x the positive angle $\tan^{-1}\frac{3}{4}$.

Exercise 27. (Review.)

1. Find the distance from the point $(-2b, b)$ to the origin, the axes making the angle 60°.

2. The axes making the angle ω, find the distance from the point $(1, -1)$ to the point $(-1, 1)$.

3. The axes making the angle ω, find the distance from the point $(0, 2)$ to the point $(3, 0)$.

Determine the distance between the following points given by polar coördinates:

4. (a, θ) and (b, ϕ).

5. (a, θ) and $(a, -\theta)$.

6. (a, θ) and $(-a, -\theta)$.

7. $(2a, 30°)$ and $(a, 60°)$.

8. Show that the polar coördinates (ρ, θ), $(-\rho, \pi+\theta)$, $(-\rho, \theta-\pi)$ all represent the same point.

9. Transform the equation $8x^2+8xy+4y^2+12x+8y+1=0$ to the new origin $(-\frac{1}{2}, -\frac{1}{2})$.

10. Transform the equation $6x^2+3y^2-24x+6=0$ to the new origin $(2, 0)$.

11. Transform the equation $\frac{x}{a}+\frac{y}{b}=1$ by changing the origin to the point $\left(\frac{a}{2}, \frac{b}{2}\right)$ and turning the axes through an angle ϕ, such that $\tan\phi=-\frac{b}{a}$.

12. Transform the equation $17x^2-16xy+17y^2=225$ to axes that bisect the axes of the old system.

Transform the following rectangular equations to polar equations, the polar axis in each case coinciding with, or being parallel to, the axis of x, and the pole being at the point whose coördinates are given:

13. $x^2 + y^2 = 8ax$; the pole $(0, 0)$.

14. $x^2 + y^2 = 8ax$; the pole $(4a, 0)$.

15. $y^2 - 6y - 5x + 9 = 0$; the pole $(\frac{5}{4}, 3)$.

16. $x^2 - y^2 - 4x - 6y - 54 = 0$; the pole $(2, -3)$.

17. $(x^2 + y^2)^2 = k^2(x^2 - y^2)$; the pole $(0, 0)$.

Transform the following polar equations to rectangular axes, the origin being at the pole and the axis of x coinciding with the polar axis:

18. $\rho^2 \sin 2\theta = 2a^2$.

19. $\rho = k \sin 2\theta$.

20. $\rho(\sin 3\theta + \cos 3\theta) = 5k \sin\theta \cos\theta$.

21. Through what angle must a set of rectangular axes be turned in order that the new axis of x may pass through the point $(5, 7)$?

22. The rectangular equation of a straight line is $Ax + By + C = 0$. Through what angle must the axes be turned in order that

(i) the term containing x may disappear?
(ii) the term containing y may disappear?

23. Deduce the following formulas for changing from one set of oblique axes to another, the origin remaining the same:

$$x = \frac{x' \sin(\omega - \alpha)}{\sin\omega} + \frac{y' \sin(\omega - \beta)}{\sin\omega},$$

$$y = \frac{x' \sin\alpha}{\sin\omega} + \frac{y' \sin\beta}{\sin\omega}.$$

NOTE. In these formulas ω denotes the angle formed by the old axes, α and β those formed by the positive directions of the new axes with the positive direction of the old axis of x.

24. From the formulas of No. 23 deduce those of § 88.

CHAPTER V.

THE PARABOLA.

THE EQUATION OF THE PARABOLA.

92. A **Parabola** is the locus of a point whose distance from a fixed point is always equal to its distance from a fixed straight line.

The fixed point is called the **Focus**; the fixed straight line, the **Directrix.**

The straight line through the focus perpendicular to the directrix is called the **Axis** of the parabola.

The intersection of the axis and the directrix is called the **Foot** of the axis.

The point in the axis halfway between the focus and the directrix is, from the definition, a point of the curve; this point is called the **Vertex** of the parabola.

The straight line joining any point of the curve to the focus is called the **Focal Radius** of the point.

A straight line passing through the focus and limited by the curve is called a **Focal Chord.**

The focal chord perpendicular to the axis is called the **Latus Rectum** or **Parameter.**

93. *To construct a parabola, having given the focus and the directrix.*

I. *By Points.* Let F (Fig. 45) be the focus, CE the directrix. Draw the axis FD, and bisect FD in A; then A is the vertex of the curve. At any point M in the axis erect a perpendicular. From F as centre, with DM as

radius, cut this perpendicular in P and Q; then P and Q are two points of the curve, for $FP=DM=$ distance of P or Q from CE. In the same way we can find as many points of the curve as we please. After a sufficient number of points has been found, we draw a continuous curve through them.

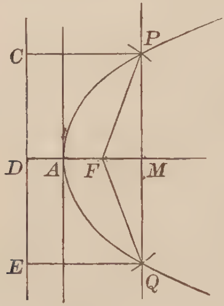


Fig. 45.

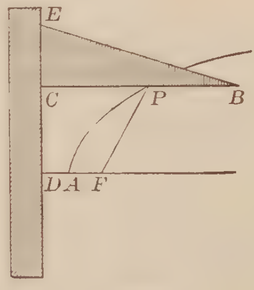


Fig. 46.

II. *By Motion.* Place a ruler so that one of its edges shall coincide with the directrix DE (Fig. 46). Then place a triangular ruler BCE with the edge CE against the edge of the first ruler. Take a string whose length is equal to BC; fasten one end at B and the other end at F. Then slide the ruler BCE along the directrix, keeping the string tightly pressed against the ruler by the point of a pencil P. The point P will trace a parabola; for during the motion we always have $PF=PC$.

94. *To find the rectangular equation of the parabola, when its axis is taken as the axis of x and its vertex as the origin.*

Let F (Fig. 45) be the focus, CE the directrix, DFX the axis, A the vertex and origin; also let $2p$ denote the known distance FD.

Let P be any point of the curve; then its coördinates are

$$AM=x, \quad MP=y.$$

Draw $PC \perp$ to CE; then by the definition of the curve

$$FP = PC \doteq DM.$$

Therefore, $\overline{FP}^2 = \overline{DM}^2.$

Now $\overline{FP}^2 = \overline{MP}^2 + \overline{FM}^2 = y^2 + (x-p)^2,$

and $\overline{DM}^2 = (x+p)^2.$

Therefore, $y^2 + (x-p)^2 = (x+p)^2.$

Whence, $\boldsymbol{y^2 = 4px.}$ [26]

This is called the *principal* equation of a parabola.

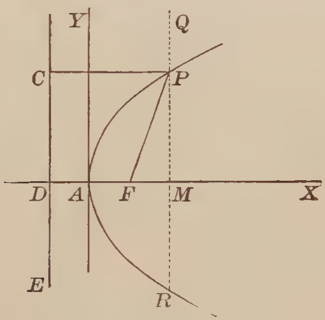


Fig. 47.

95. Since y^2 and p in equation [26] are positive, x must always be positive; therefore, the curve lies wholly on the positive side of the axis of y.

An examination of equation [26] shows that the curve, (i) passes through the origin, (ii) is symmetrical with respect to the axis of x, (iii) extends towards the right without limit, (iv) recedes from the axis of x without limit.

96. *Any point (h, k) is outside, on, or inside the parabola $y^2 = 4px$, according as $k^2 - 4ph$ is positive, zero, or negative.*

Let Q be the point (h, k), and let its ordinate meet the curve in P.

If $k^2 - 4ph = 0$, the point (h, k) satisfies equation [26] and therefore Q coincides with P.

If $k^2 - 4ph$ is positive, or $k^2 > 4ph$, then, since $\overline{PM}^2 = 4ph$, we have $\overline{QM}^2 > \overline{PM}^2$, or $QM > PM$; hence, Q is outside the curve.

If $k^2 - 4ph$ is negative, we may prove similarly that Q must be inside the curve.

97. If $x = p$, $y = \pm 2p$. But these two values of y make up the latus rectum. Hence, the *latus rectum* $= 4p$.

COR. From the equation $y^2 = 4px$, it follows that

$$x : y = y : 4p;$$

that is, *the latus rectum is a third proportional to any abscissa and its corresponding ordinate.*

98. If (x_1, y_1) and (x_2, y_2) are any two points on the parabola, we have

$$y_1^2 = 4px_1, \quad y_2^2 = 4px_2.$$

Hence, $$y_1^2 : y_2^2 = x_1 : x_2;$$

that is, *the squares of the ordinates of any two points on the parabola are to each other as their abscissas.*

99. *To find the points in which the straight line $y = mx + c$ meets the parabola $y^2 = 4px$.*

Regarding these equations as simultaneous, and eliminating x, we have

$$y^2 = 4p\frac{y - c}{m}. \tag{1}$$

Whence, $$y = \frac{2p}{m} \pm \frac{2p}{m}\sqrt{\frac{p - mc}{p}}. \tag{2}$$

From (2) it follows that $y = mx + c$ has two distinct, two coincident, or no points in common with $y^2 = 4px$, according as $p - mc >, =$, or < 0.

COR. If $p - mc = 0$, or $c = p \div m$, $y = mx + c$ will be a tangent; that is, $$y = mx + \frac{p}{m} \tag{3}$$

is a tangent to $y^2 = 4px$ in terms of its slope.

Exercise 28.

1. Show that the distance of any point of the parabola $y^2 = 4px$ from the focus is equal to $p + x$.

2. Find the equation of a parabola, taking as axes the axis of the curve and the directrix.

3. Find the equation of a parabola, taking the axis of the curve as the axis of x and the focus as the origin.

4. The distance from the focus of a parabola to the directrix $= 5$. Write its equation,

(i) if the origin is taken at the vertex.
(ii) if the origin is taken at the focus.
(iii) if the axis and directrix are taken as axes.

5. The distance from the focus to the vertex of a parabola is 4. Write its equations for the three cases enumerated in No. 4.

6. For what point of the parabola $y^2 = 18x$ is the ordinate equal to three times the abscissa?

7. Find the latus rectum of the following parabolas:

$$y^2 = 6x, \quad y^2 = 15x, \quad by^2 = ax.$$

Find the points common to the following parabolas and straight lines:

8. $y^2 = 9x, \quad 3x - 7y + 30 = 0.$

9. $y^2 = 3x, \quad x - 4y + 12 = 0.$

10. $y^2 = 4x, \quad x = 9, \quad x = 0, \quad x = -2.$

11. $y^2 = 4x, \quad y = 6, \quad y = -8.$

12. What must be the value of p in order that the parabola $y^2 = 4px$ may pass through the point $(9, -12)$?

13. For what point of the parabola $y^2 = 32x$ is the ordinate equal to 4 times the abscissa?

14. The equation of a parabola is $y^2 = 8x$. What is the equation of (i) its axis, (ii) its directrix, (iii) its latus rectum, (iv) a focal chord through the point whose abscissa $= 8$, (v) a chord passing through the vertex and the negative end of the latus rectum?

15. The equation of a parabola is $y^2 = 16x$. Find the equation of (i) a chord through the points whose abscissas are 4 and 9, and ordinates positive; (ii) the circle passing through the vertex and the ends of the latus rectum.

16. If the distance of a point from the focus of the parabola $y^2 = 4px$ is equal to the latus rectum, what is the abscissa of the point?

17. In the parabola $y^2 = 4px$ an equilateral triangle is inscribed so that one vertex is at the origin. What is the length of one of its sides?

18. A double ordinate of a parabola $= 8p$. Prove that straight lines drawn from its ends to the vertex are perpendicular to each other.

Explain how to construct a parabola, having given:

19. The directrix and the vertex.

20. The focus and the vertex.

21. The axis, vertex, and latus rectum.

22. The axis, vertex, and a point of the curve.

23. The axis, focus, and latus rectum.

24. Determine, as regards size and position, the relations of the following parabolas:

(i) $y^2 = 4px$, (ii) $y^2 = -4px$, (iii) $x^2 = 4py$, (iv) $x^2 = -4py$.

Tangents and Normals.

100. *To find the equation of the tangent and of the normal to the parabola $y^2 = 4px$ at any point (x_1, y_1).*

Let (x_1, y_1), (x_2, y_2) be any two points on the parabola; then the equation of the secant through them is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{1}$$

Since (x_1, y_1) and (x_2, y_2) are on the curve $y^2 = 4px$, we have

$$y_1^2 = 4px_1, \; y_2^2 = 4px_2.$$

Whence, $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4p}{y_2 + y_1}.$$

By substituting in (1), the equation of the secant becomes

$$\frac{y - y_1}{x - x_1} = \frac{4p}{y_2 + y_1}. \tag{2}$$

Now, if (x_2, y_2) is made to coincide with (x_1, y_1), (2) becomes the equation of the tangent at (x_1, y_1). Putting $y_2 = y_1$, clearing of fractions, and remembering that $y_1^2 = 4px_1$, we obtain as the equation of the tangent at (x_1, y_1),

$$\boldsymbol{y_1 y = 2p(x + x_1)}. \qquad [27]$$

The normal passes through (x_1, y_1), and is perpendicular to the tangent; hence, its equation is, by [27] and § 46,

$$\boldsymbol{y - y_1 = -\frac{y_1}{2p}(x - x_1)}. \qquad [28]$$

101. If in [27] we put $y = 0$, we obtain

$$x = -x_1, \text{ or } TA = AM \text{ (Fig. 48)}.$$

Therefore, *the subtangent is bisected at the vertex.*

If in [28] we put $y = 0$, we obtain

$$x = x_1 + 2p, \text{ or } x - x_1 [= MN] = 2p \text{ (Fig. 48)}.$$

Hence, *the subnormal is constant and equal to the semi-latus rectum.*

Cor. These properties furnish simple methods for drawing tangents to the parabola. Thus, to draw a tangent to the parabola at P (Fig. 48), draw the ordinate PM, lay off $AT=AM$, and draw PT, which will be the tangent at P by § 101. Or lay off $MN=FD$, and draw PN; then PT perpendicular to PN at P will be the tangent at P.

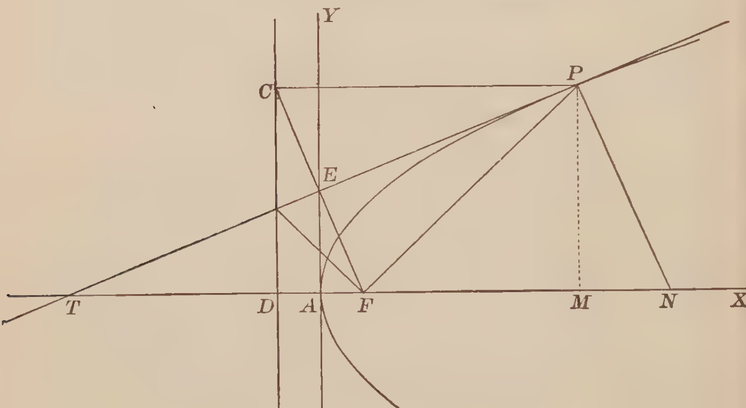


Fig. 48.

102. In the triangle FPT (Fig. 48) we have

$$FT=FA+AT=p+x,$$

$$FP=PC=DM=DA+AM=p+x.$$

Therefore, $FT=FP.$

Hence, the angle

$$FPT=PTF=TPC, \text{ or}$$

The tangent to a parabola at any point makes equal angles with the axis of the curve and the focal radius to the point of contact.

Exercise 29.

1. The normal to a parabola at any point bisects the angle between the focal radius and the line drawn through the point parallel to the axis.

NOTE. The use of parabolic reflectors depends on this property. A ray of light issuing from the focus and falling on the reflector is reflected in a line parallel to the axis of the reflector.

2. Explain how to draw a tangent and a normal to a given parabola at a given point.

3. Prove that FC (Fig. 48) is perpendicular to PT.

4. Prove that the tangent $y = mx + \frac{p}{m}$ touches the parabola $y^2 = 4px$ at the point $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$.

5. Prove that the equation of a normal to the parabola $y^2 = 4px$ in terms of its slope is $y = mx - mp(2 + m^2)$.

6. What are the equations of a tangent and a normal to the parabola $y^2 = 5x$, that pass through the point whose abscissa is 20 and ordinate positive?

7. What are the equations of the tangents and the normals to the parabola $y^2 = 12x$, drawn through the ends of the latus rectum? Find the area of the figure they enclose.

8. Through the point on the parabola $y^2 = 10x$ whose abscissa is 7 and ordinate positive a tangent and a normal are drawn. Find the lengths of the tangent, the normal, the subtangent, and the subnormal.

9. A tangent to the parabola $y^2 = 20x$ makes with the axis of x an angle of 45°. Determine the point of contact.

10. Show that the focus F (Fig. 48) is equidistant from the points P, T, N. What easy way of drawing a tangent and a normal is suggested by this theorem?

11. If F is the focus of a parabola, and Q, R denote the points in which a tangent cuts the directrix and the latus rectum produced, prove that $FQ = FR$.

12. Prove that the tangents drawn through the ends of the latus rectum are perpendicular to each other.

13. Find the distances of the vertex and the focus from the tangent $y = mx + \frac{p}{m}$.

14. Find the point of intersection of the tangents to the parabola $y^2 = 4px$ at the points (x_1, y_1), (x_2, y_2).

15. A tangent to the parabola $y^2 = 4px$ cuts equal intercepts on the axes. What is its equation? What is the point of contact? What is the value of each intercept?

16. Through what point in the axis of x must tangents to the parabola $y^2 = 4px$ be drawn in order that they may form with the tangent, through the vertex, an equilateral triangle?

17. For what point of the parabola $y^2 = 4px$ is the normal equal to twice the subtangent?

18. For what point of the parabola $y^2 = 4px$ is the normal equal to the difference between the subtangent and the subnormal?

19. Find the equation of the tangent to the parabola $y^2 = 5x$ parallel to the straight line $3x - 2y + 7 = 0$. Also find the point of contact.

20. Find the equation of the straight line that touches the parabola $y^2 = 12x$ and makes an angle of $45°$ with the line $y = 3x - 4$. Also find the point of contact.

21. Find the equation of a straight line that touches the parabola $y^2 = 16x$ and passes through the point $(-4, 8)$.

22. If a normal to a parabola for the point P meets the curve again in the point Q, find the length of PQ.

23. Prove by the secant method that the equation of a tangent to the parabola $y^2 = 4px - 4p^2$, at the point (x_1, y_1) is

$$y_1 y = 2p(x + x_1) - 4p^2.$$

24. Find the equations of the tangents and normals to the parabola $y^2 - 8x - 6y - 63 = 0$, for the points whose common abscissa $= -1$.

25. What are the general equations of tangents to the following parabolas:

(i) $y^2 = -4px$? (ii) $x^2 = 4py$? (iii) $x^2 = -4py$?

Exercise 30. (Review.)

NOTE. If not otherwise specified, the axis of the parabola and the tangent at the vertex are to be assumed as the axes of coördinates.

What is the equation of a parabola:

1. If the axis and directrix are taken as axes, and the focus is the point (12, 0)?

2. If the axis and tangent at the vertex are the two axes, and (25, 20) is a point on the curve?

3. If the same axes are taken, and the focus is the point $(-4\frac{1}{4}, 0)$?

4. If the axis is parallel to the axis of x, the vertex is the point $(5, -3)$, and the latus rectum $= 5\frac{1}{2}$?

5. If the axis is the line $y = -7$, the abscissa of the vertex $= 3$, and one point is $(4, -5)$?

6. If the curve passes through the points (0, 0), (3, 2), $(3, -2)$?

7. If the curve passes through the points (0, 0), (3, 2), $(-3, 2)$?

8. What is the latus rectum of the parabola $2y^2 = 3x$? What is the equation of its directrix, and of the focal chords passing through the points whose abscissa $= 6$?

9. Describe the change of form which the parabola $y^2 = 4px$ undergoes as we suppose p to diminish without limit.

10. Find the intercepts of the parabola

$$y^2 + 4x - 6y - 16 = 0.$$

11. One vertex of an equilateral triangle coincides with the focus, and the others lie on the parabola $y^2 = 4px$. Find the length of one side.

12. The latus rectum of a parabola $= 8$; find

(i) Equation of a tangent through its positive end.

(ii) Distance from the focus to this tangent.

(iii) Equation of the normal at this point.

13. What is the equation of the chord passing through the two points of the parabola $y^2 = 8x$ for which $x = 2$, $y > 0$, and $x = 18$, $y < 0$?

14. Find the equation of the chord of the parabola $y^2 = 4px$ that is bisected at a given point (x_1, y_1).

15. In what points does the line $x + y = 12$ meet the parabola $y^2 + 2x - 12y + 16 = 0$?

16. In what points does the line $3y = 2x + 8$ meet the parabola $y^2 - 4x - 8y + 24 = 0$?

17. Find the equations of tangents from the origin to the parabola $(y - b)^2 = 4p(x - a)$.

18. Describe the position of the parabola $y^2 + 2x + 4 = 0$ with respect to the axes, and determine its latus rectum, vertex, focus, and directrix.

19. What is the distance from the origin to a normal drawn through the end of the latus rectum of the parabola

$$y^2 = 4a(x - a)?$$

Find the equation of the parabola:

20. If the equation of a tangent is $4y = 3x - 12$.

21. If a focal radius $= 10$, and its equation is $3y = 4x - 8$.

22. If for a point of the curve the focal radius $= r$, and the length of the tangent $= t$.

23. If for a point of the curve the focal radius $= r$, and the length of the normal $= n$.

24. If for a point of the curve the length of the tangent $= t$, and the length of the normal $= n$.

25. If for a point of the curve the focal radius $= r$, and the subtangent $= s$.

26. Two parabolas have the same vertex, and the same latus rectum $4p$, but their axes are $\perp$ to each other. What is the length of their common chord?

27. Through the three points of the parabola $y^2 = 12x$, whose ordinates are 2, 3, 6, tangents are drawn. Show that the circle circumscribed about the triangle formed by the tangents passes through the focus.

28. A tangent to the parabola $y^2 = 4px$ makes the angle $30°$ with the axis of x. At what point does it cut the axis?

29. For what point of the parabola $y^2 = 4px$ is the length of the tangent equal to 4 times the abscissa of the point of contact?

30. The product of the tangent and normal is equal to twice the square of the ordinate of the point of contact. Find the point of contact and the inclination of the tangent to the axis of x.

31. Two tangents to a parabola are perpendicular to each other. Find the product of their subtangents.

32. Prove that the circle described on a focal radius as diameter touches the tangent drawn through the vertex.

33. Prove that the circle described on a focal chord as diameter touches the directrix.

Find the locus of the middle points :

34. Of all the ordinates of a parabola.

35. Of all the focal radii.

36. Of all the focal chords.

37. Of all chords passing through the vertex.

38. Of all chords that meet at the foot of the axis.

Two tangents to the parabola $y^2 = 4px$ make the angles θ, θ' with the axis of x; find the locus of their intersection :

39. If $\cot\theta + \cot\theta' = k$.

40. If $\cot\theta - \cot\theta' = k$.

41. If $\tan\theta \tan\theta' = k$.

42. If $\sin\theta \sin\theta' = k$.

43. Find the locus of the centre of a circle that passes through a given point and touches a given straight line.

SUPPLEMENTARY PROPOSITIONS.

103. *Two distinct, two coincident, or no real tangents can be drawn to a parabola from any point* (h, k), *according as the point is without, on, or within the curve.*

Let the tangent $y = mx + \frac{p}{m}$ pass through the point (h, k)

then, $$k = mh + \frac{p}{m},$$

or $$hm^2 - km + p = 0.$$

Whence, $$m = \frac{k \pm \sqrt{k^2 - 4ph}}{2h}.$$

These values of m are real and unequal, real and equal, or imaginary, according as $k^2 - 4ph >, =,$ or < 0; that is, according as (h, k) is without, on, or within the parabola; hence the proposition (§ 96).

104. *To find the equation of the chord of contact of two tangents drawn from any external point* (h, k) *to the parabola* $y^2 = 4px$.

Let (x_1, y_1) and (x_2, y_2) be the points of contact; then the equations of the tangents are

$$y_1 y = 2p(x + x_1),$$
$$y_2 y = 2p(x + x_2).$$

Since (h, k) is in both these lines, we have

$$ky_1 = 2p(x_1 + h), \tag{1}$$
$$ky_2 = 2p(x_2 + h). \tag{2}$$

From equations (1) and (2) we see that both the points (x_1, y_1) and (x_2, y_2) lie in the straight line whose equation is

$$ky = 2p(x + h). \tag{3}$$

Hence, (3) is the equation required.

105. *To find the equation of the polar of the pole* (h, k) *with regard to the parabola* $y^2 = 4px$.

Let P be the fixed point (h, k), PQR one position of the revolving chord, and let the tangents at Q and R intersect in $P_1(x_1, y_1)$; it is required to find the locus of P_1, as the chord turns about P.

Since PR is the chord of contact of tangents drawn from the point $P_1(x_1, y_1)$, its equation is (§ 104)

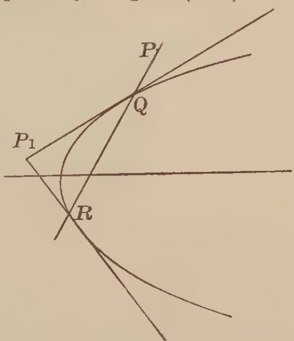


Fig. 49.

$$y_1 y = 2p(x + x_1). \tag{1}$$

Since (1) passes through (h, k) we have

$$y_1 k = 2p(h + x_1). \tag{2}$$

But (x_1, y_1) is any point on the required locus, and by (2) its coördinates satisfy the equation

$$ky = 2p(x + h). \tag{3}$$

Hence, (3) is the required equation, and the polar is a straight line.

Cor. When the pole (h, k) is on the curve, the polar is evidently a tangent at (h, k); when the pole (h, k) is without the curve, the polar is the chord of contact of tangents from (h, k). Thus the *tangent* and *chord of contact* are particular cases of the *polar*.

The Proposition of § 74 may be proved for poles and polars with respect to a parabola.

106. *To find the locus of the middle points of parallel chords in the parabola $y^2 = 4px$.*

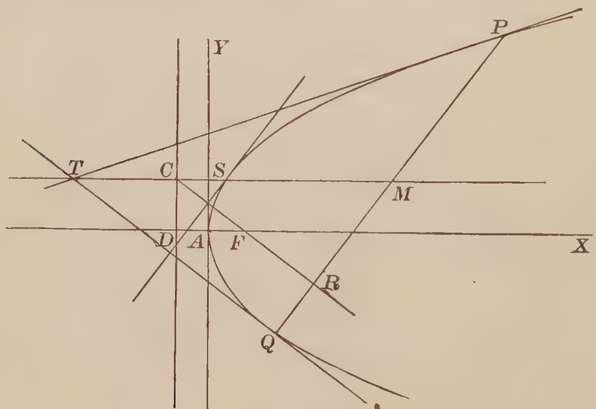


Fig. 50.

Let any one of the chords PQ (Fig. 50) be $y = mx + c$, and let it meet the curve in the points (x_1, y_1), (x_2, y_2).

Then (§ 100), $$m = \frac{4p}{y_1 + y_2}. \qquad (1)$$

Let M (x, y) be the middle point of PQ; then $2y = y_1 + y_2$. By substitution in (1) we obtain

$$m = \frac{2p}{y} \text{ or } y = \frac{2p}{m}, \qquad (2)$$

a relation that holds true for all the parallel chords, because m is the same for all the chords. The required locus, therefore, is represented by (2), and is a straight line parallel to the axis of x, and called a *diameter* of the parabola. Hence,

Every diameter of a parabola is a straight line parallel to its axis.

Every straight line parallel to the axis is a diameter; for m, and, therefore $\frac{2p}{m}$, may have any value whatever.

107. Let the diameter through M meet the curve at S, and conceive the straight line PQ to move parallel to itself till P and Q coincide at S; then the straight line becomes the tangent at S; therefore,

The tangent drawn through the extremity of a diameter is parallel to the chords of that diameter.

108. From the focus F draw $FC \perp$ to PQ, and let FC meet the directrix in the point C. If θ denotes the angle which the chord PQ makes with the axis of x, it easily follows that $DCF = \theta$; then we have

$$CD = FD \cot \theta = \frac{2p}{m};$$

hence, by (2) § 106,

The perpendicular from the focus to a chord meets the diameter of the chord in the directrix.

Moreover, since DS (Fig. 50) is parallel to QP, *the perpendicular from the focus to a tangent and the diameter through the point of contact meet in the directrix.*

109. Let the tangents drawn through P and Q meet in the point T. Regarding their equations,

$$y_1 y = 2p(x + x_1),$$
$$y_2 y = 2p(x + x_2),$$

as simultaneous, we obtain for the value of the ordinate of T

$$y = \frac{2p(x_2 - x_1)}{y_2 - y_1} = \frac{2p}{m}.$$ Hence,

Tangents drawn through the ends of a chord meet in the diameter of the chord.

110. *To find the locus of the foot of a perpendicular from the focus to a tangent.*

Let the equation of a tangent be

$$y = mx + \frac{p}{m}.$$

Then the equation of the perpendicular will be

$$y = -\frac{x}{m} + \frac{p}{m}.$$

Since these two lines have the same intercept on the axis of y, they meet in that axis; hence, the tangent through the vertex is the required locus.

111. Since $FP = PC$ (Fig. 48) and angle $EPC = EPF$, therefore the tangent PT is perpendicular to FC at its middle point, and every point in it is equally distant from F and C.

112. *Tangents at right angles intersect in the directrix.*

Let the equation of one tangent be

$$y = mx + \frac{p}{m} \tag{1}$$

Then the equation of the other is

$$y = -\frac{x}{m} - mp. \tag{2}$$

Subtracting (2) from (1) we obtain for their common point

$$(x+p)\left(m+\frac{1}{m}\right)=0.$$

But $m+\dfrac{1}{m}$ cannot be zero; hence, $x+p=0$, or $x=-p$, which is the equation of the directrix.

113. The polar of the focus $(p, 0)$ is

$$0=2p\,(x+p), \text{ or } x=-p.$$

Hence, *the polar of the focus is the directrix, or tangents at the ends of a focal chord intersect in the directrix.*

COR. From this result and § 112 it follows that *tangents through the ends of a focal chord intersect at right angles.*

114. *To find the equation of a parabola referred to any diameter and the tangent through its extremity as axes.*

Transform the equation $y^2=4px$ to the diameter SX' (Fig. 51) and the tangent through S as new axes. Let m be the slope of the tangent, θ the angle which the tangent makes with the diameter; then $m=\tan\theta$.

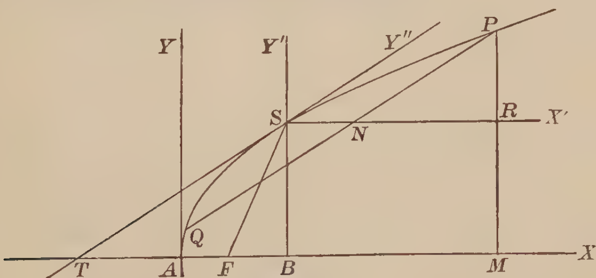


Fig. 51.

First transform to new parallel axes through S.

Now, by § 106, $BS=2p\div m$; hence, from $y^2=4px$ we obtain $AB=p\div m^2$. Therefore, the new equation is

$$\left(y+\frac{2p}{m}\right)^2=4p\left(x+\frac{p}{m^2}\right),$$

or

$$my^2+4py=4pmx. \tag{1}$$

Now retain the axis of x, and turn the axis of y till it coincides with the tangent at S; then for any point P we have

The old $x = SR$. The new $x = SN$.

The old $y = RP$. The new $y = NP$.

Now it is easily seen from Fig. 52 that

$$SR = SN + NP \cos\theta, \qquad RP = NP \sin\theta.$$

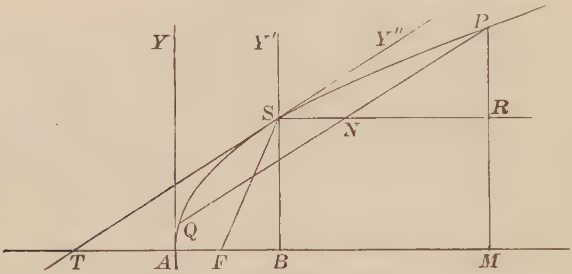


Fig. 52.

Therefore, equation (1) is transformed to the new system by writing $x + y\cos\theta$ in place of x, and $y\sin\theta$ in place of y. Making this substitution, remembering that $m = \tan\theta$, and reducing, we obtain

$$y^2 = \frac{4p}{\sin^2\theta}\, x, \tag{2}$$

an equation of the same form as $y^2 = 4px$.

Join S to the focus F, and denote FS by p'; then

$$p' = p + AB = p + \frac{p}{m^2} = \frac{p(1+m^2)}{m^2} = \frac{p}{\sin^2\theta}.$$

Therefore, equation (2) may be more simply written

$$y^2 = 4p'x, \tag{3}$$

where p' is the distance of the origin from the focus. It is easy to see that this equation includes the case when the axes are the axis of the curve and the tangent at the vertex.

The quantity $4p'$ is called the **Parameter** of the diameter passing through S. When the diameter is the axis of the curve, $4p'$ is called the **Principal Parameter**.

COR. Let the equation of a parabola referred to any diameter, and the tangent at the end of that diameter as axes, be $y^2 = 4p'x$. Since the investigations in §§ 99, 100 hold good whether the axes are at right angles or not, it follows immediately that the straight line $y = mx + \frac{p'}{m}$ will touch the parabola for all values of m, and that the equation of the tangent at any point (x_1, y_1) is $y_1y = 2p'(x + x_1)$.

115. *To find the polar equation of the parabola, the focus being the pole.*

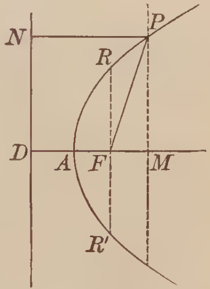


Fig. 53.

Let P be any point (ρ, θ) of the curve; then

$$\rho = FP = NP = DM = 2p + FM$$

$$= 2p + \rho \cos \theta.$$

$$\therefore \rho = \frac{2p}{1 - \cos \theta}. \qquad [29]$$

Discussion of [29]:

Since $\cos \theta$ cannot exceed $+1$, ρ is positive for all values of θ.

If $\theta = 0$, $\cos \theta = 1$, and $\rho = \infty$.

This shows that the axis of the parabola does not cut the curve to the right of the focus.

If $\theta = \frac{1}{2}\pi$, $\cos \theta = 0$; $\therefore \rho = 2p =$ semi-latus rectum.

If $\theta = \pi$, $\cos \theta = -1$; $\therefore \rho = p = FA$.

If $\theta = \frac{3}{2}\pi$, $\cos \theta = 0$; $\therefore \rho = 2p = FR'$.

If $\theta = 2\pi$, $\cos \theta = 1$; $\therefore \rho = \infty$.

As θ increases from zero to π, ρ decreases from ∞ to p. As θ increases from π to 2π, ρ increases from p to ∞.

Exercise 31.

1. Given a parabola, to draw its axis (§106).

2. Prove that the perpendicular dropped from any point of the directrix to the polar of the point passes through the focus.

3. To find by construction the pole of a focal chord.

4. Prove that through any point *three* normals can be drawn to a parabola.

5. Tangents are drawn through the ends of a chord. Prove that the part of the corresponding diameter contained between the chord and the intersection of the tangents is bisected by the curve.

6. Focal radii are drawn to two points of a parabola, and tangents are then drawn through these points. Prove that the angle between the tangents is equal to half the angle between the focal lines.

7. Show that if the vertex is taken as pole, the polar equation of a parabola is

$$\rho = \frac{4p\cos\theta}{\sin^2\theta}.$$

8. Explain how tangents to a parabola may be drawn from an exterior point (§ 102).

9. Having given a parabola, how would you find its axis, directrix, focus, and latus rectum?

10. From the point $(-2, 5)$ tangents are drawn to the parabola $y^2 = 6x$. What is the equation of the chord of contact?

11. The general equation of a system of parallel chords in the parabola $7y^2 = 25x$ is $4x - 7y + k = 0$. What is the equation of the corresponding diameter?

12. In the parabola $y^2 = 13x$, what is the equation of the ordinates of the diameter $y + 11 = 0$?

13. In the parabola $y^2 = 6x$, what chord is bisected at the point (4, 3)?

14. Given the parabola $y^2 = 4px$; find the equation of the chord that passes through the vertex and is bisected by the diameter $y = a$. How can this chord be constructed?

15. The latus rectum of a parabola $= 16$. What is the equation of the curve if a diameter at the distance 12 from the focus, and the tangent through its extremity, are taken as axes?

16. Show that the equation of that chord of the parabola $y^2 = 4px$ which is bisected at the point (h, k) is

$$k(y - k) = 2p(x - h).$$

17. Prove that the parameter of any diameter is equal to the focal chord of that diameter.

18. Prove that the locus of $y^2 - 8y - 6x + 28 = 0$ is a parabola whose axis is parallel to the axis of x; and determine the latus rectum, the vertex, the focus, the axis, and the directrix.

19. Prove that in general the locus of $y^2 + Ax + By + C = 0$ is a parabola whose axis is parallel to the axis of x; and determine its latus rectum, vertex, and axis.

20. Prove that in general the locus of $x^2 + Ax + By + C = 0$ is a parabola whose axis is parallel to the axis of y; and determine its latus rectum, vertex, and axis.

21. Find the locus of the centres of circles that touch a given circle and also a given straight line.

22. The area and base of a triangle being given, find the locus of the intersection of perpendiculars dropped from the ends of the base to the opposite sides.

CHAPTER VI.

THE ELLIPSE.

Simple Properties of the Ellipse.

116. The **Ellipse** is the locus of a point, the sum of whose distances from two fixed points is constant.

The fixed points are called **Foci**; and the distance from any point of the curve to a focus is called a **Focal Radius.**

The constant sum is denoted by $2a$, and the distance between the foci by $2c$.

The fraction $\frac{c}{a}$ is called the **Eccentricity,** and is represented by the letter e. Therefore, $c = ae$.

In the ellipse $a > c$; that is, $e < 1$.

If $a = c$, the locus is simply the limited straight line joining the foci.

If $a < c$, from the definition it is clear that there is no locus.

117. *To construct an ellipse, having given the foci and the constant sum* $2a$.

I. *By Motion.* Fix pins in the paper at the foci. Tie a string to them, making the length of the string exactly equal to $2a$. Then press a pencil against the string so as to make it tense, and move the pencil, keeping the string constantly stretched. The point of the pencil will trace the required ellipse; for in every position the sum of the distances from the point of the pencil to the foci is equal to the length of the string.

II. *By Points.* Let F, F' be the foci; then $FF' = 2c$.

Bisect FF' at O, and from O lay off $OA = OA' = a$.

Then $A'A = 2a$, $F'A' = FA$,

$$A'F + A'F' = A'F + FA = 2a,$$
$$AF' + AF = AF' + F'A' = 2a.$$

Therefore, A and A' are points of the curve.

Between F and F' mark any point X; then describe two arcs, one with F as centre and AX as radius, the other with F' as centre and $A'X$ as radius; the intersections P, Q of

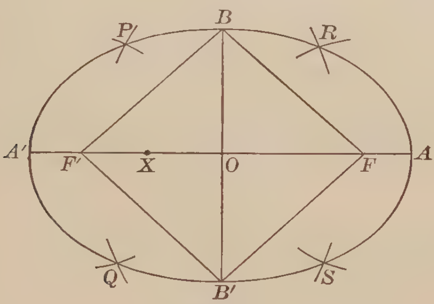


Fig. 54.

these arcs are points of the curve. By merely interchanging the radii, two more points, R, S, may be found.

After a sufficient number of points has been obtained, draw a continuous curve through them.

118. The line AA' is the **Transverse** or **Major Axis,** A, A' the **Vertices,** and O the **Centre** of the curve.

The line BB', perpendicular to the major axis at O, is the **Conjugate** or **Minor Axis;** its length is denoted by $2b$.

Show that B and B' are equidistant from the foci, that $BF = a$, that $BO = b$, and that $a^2 = b^2 + c^2 = b^2 + a^2e^2$.

119. *To find the equation of the ellipse, having given the foci and the constant sum 2a.*

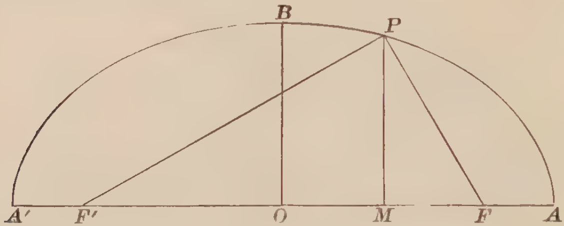


Fig. 55.

Take the line AA' (Fig. 55), passing through the foci, as the axis of x, and the point O, halfway between the foci, as origin. Let P be any point (x, y) of the curve, and let r, r' denote the focal radii of P. Then from the definition of the curve, and from the right triangles $F'PM$, FPM,

$$r'^2 = y^2 + (c + x)^2, \qquad (1)$$

$$r^2 = y^2 + (c - x)^2. \qquad (2)$$

By addition, $$r'^2 + r^2 = 2(x^2 + y^2 + c^2). \qquad (3)$$

By subtraction, $$r'^2 - r^2 = 4cx. \qquad (4)$$

But $$r' + r = 2a. \qquad (5)$$

By division, $$r' - r = \frac{2cx}{a}. \qquad (6)$$

By subtraction, $$r = a - \frac{cx}{a} = [a - ex]. \qquad (7)$$

By addition, $$r' = a + \frac{cx}{a} = [a + ex]. \qquad (8)$$

$$r'^2 + r^2 = 2\left(a^2 + \frac{c^2x^2}{a^2}\right)$$

Substitute in (3) remembering that $b^2 = a^2 - c^2$ (§ 118).

Then $$b^2x^2 + a^2y^2 = a^2b^2,$$

or $$\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.} \qquad [30]$$

COR. If the transverse axis is on the axis of y, and the conjugate on the axis of x, the equation of the ellipse is

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1. \tag{10}$$

120. *To trace the form of the ellipse from its equation.*

The intercepts on the axis of x are $+a$ and $-a$; on the axis of y, $+b$ and $-b$.

Only the squares of the variables x and y appear in the equation; hence, if it is satisfied by a point (x, y), it will also be satisfied by the points $(x, -y)$, $(-x, y)$, $(-x, -y)$. Therefore, we infer that

(i) *The curve is symmetrical with respect to the axis of x.*

(ii) *The curve is symmetrical with respect to the axis of y.*

(iii) *The curve is symmetrical with respect to the centre O, which bisects every chord passing through it.* This explains why O is called the centre.

Since the sum of $\left(\frac{x}{a}\right)^2$ and $\left(\frac{y}{b}\right)^2$ is 1, neither of these squares can exceed 1; therefore, the maximum value of x is $+a$, and the minimum value $-a$, while the corresponding values of y are $+b$ and $-b$. Therefore, the curve is wholly contained within the rectangle whose sides are $x=\pm a$, $y=\pm b$.

121. *To trace the changes in the form of the ellipse when the semi-axes are supposed to change.*

Let a be regarded as a constant. and b as a variable.

(i) Suppose b to increase. Then c decreases (since $c^2=a^2-b^2$), e decreases, the foci approach the centre, and the ellipse approaches the circle.

(ii) Let $b=a$. Then $c=0$, $e=0$, the foci coincide with the centre, the ellipse becomes a circle of radius a, and equation [30] becomes the equation of the circle,

$$x^2+y^2=a^2.$$

(iii) If we suppose b to decrease to 0 (a remaining constant), c will increase to a, e will increase to 1, while the curve will approach, and finally coincide with, the major axis, its equation at the same time becoming $y = 0$.

122. Let (x_1, y_1) and (x_2, y_2) be any two points on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$; then we have

$$y_1^2 = \frac{b^2}{a^2}(a^2 - x_1^2), \qquad y_2^2 = \frac{b^2}{a^2}(a^2 - x_2^2).$$

Dividing and factoring, we have

$$y_1^2 : y_2^2 :: (a - x_1)(a + x_1) : (a - x_2)(a + x_2).$$

This is, *the squares of any two ordinates of the ellipse are to each other as the products of the segments into which they divide the major axis.*

123. It follows from § 119 that a point (h, k) is *on* the ellipse represented by the equation [30], provided

$$\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1 = 0.$$

It may be shown by reasoning similar to that employed in § 96 that the point (h, k) is *outside* or *inside* the curve, according as $\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$ is *positive* or *negative*.

124. If A, B, C all have the same sign, every equation of the form

$$Ax^2 + By^2 = C \tag{1}$$

may be reduced to the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ or } \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

Hence, every equation of the form of (1) represents an ellipse whose semi-axes are $\sqrt{\frac{C}{A}}$ and $\sqrt{\frac{C}{B}}$. The transverse axis lies on the axis of x or the axis of y, according as A is *less* than or *greater* than B.

125. The chord passing through either focus perpendicular to the major axis is called the **Latus Rectum** or **Parameter.**

To find its length, put $x=c$ in the equation of the ellipse.

Then, $$y^2=\frac{b^2}{a^2}(a^2-c^2)=\frac{b^4}{a^2},\ y=\pm\frac{b^2}{a}.$$

Therefore, the latus rectum $=\dfrac{2b^2}{a}=\left[\dfrac{4b^2}{2a}\right].$

Forming a proportion from this equation, we have

$$2a:2b::2b:\text{latus rectum};$$

that is, *the latus rectum is a third proportional to the major and minor axes.*

126. The circle having for diameter the major axis of the ellipse is called the **Auxiliary Circle;** its equation is

$$x^2+y^2=a^2.$$

The circle having for diameter the minor axis is called the **Minor Auxiliary Circle;** its equation is

$$x^2+y^2=b^2.$$

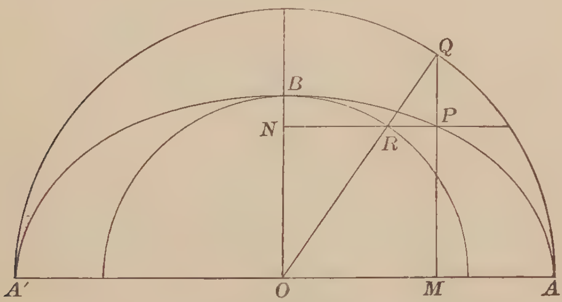


Fig. 56.

If P (Fig. 56) is any point of an ellipse, and the ordinate MP produced meets the auxiliary circle in Q, the point Q is said to *correspond* to the point P.

The angle QOM is called the **Eccentric Angle** of the point P, and is denoted by the letter ϕ.

127. Let y, y' represent the ordinates of points in an ellipse and the auxiliary circle respectively, corresponding to the same abscissa x. Then from the equations of the two curves we have

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}, \quad y' = \pm \sqrt{a^2 - x^2}.$$

Whence, $$y : y' = b : a,$$

or, *the ordinates of the ellipse and the auxiliary circle, corresponding to a common abscissa, are to each other in the constant ratio of the semi-minor and semi-major axes of the ellipse.*

128. The principle of § 127 furnishes the following easy method of constructing an ellipse by points when its axes are given:

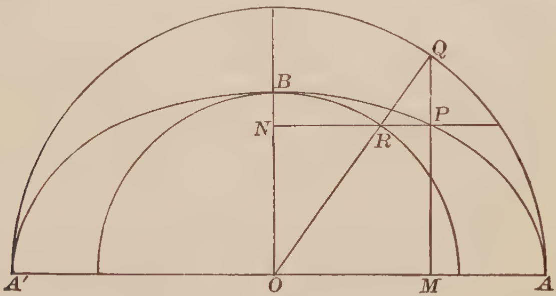


Fig. 57.

Construct both the major and minor auxiliary circles (Fig. 57); draw any radius cutting the circles in R and Q; through Q draw a line parallel to BO, and through R draw a line parallel to OA; the intersection P of these lines is a point on the ellipse. For we have

$$MP : MQ = OR : OQ,$$

or $$MP : y' = b : a.$$

From this proportion and that in § 127, we have $MP = y$; hence, P is a point on the ellipse. In like manner any number of points may be found.

COR. From Fig. 57, we have

$$\left.\begin{aligned} x &= OM = OQ \cos\phi = a\cos\phi, \\ y &= MP = ON = OR \sin\phi = b\sin\phi. \end{aligned}\right\} \quad (1)$$

Equations (1), which express the coördinates of any point of the ellipse in terms of its eccentric angle, may be used as the equations of the ellipse. To obtain from them the common equation, we have

$$\frac{x}{a} = \cos\phi, \text{ and } \frac{y}{b} = \sin\phi.$$

Therefore, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \cos^2\phi + \sin^2\phi = 1.$

129. *To find the area of an ellipse.*

Divide the semi-major axis OA' (Fig. 58) into any number of equal parts, through any two adjacent points of division M, N erect ordinates, and let the ordinate through M meet

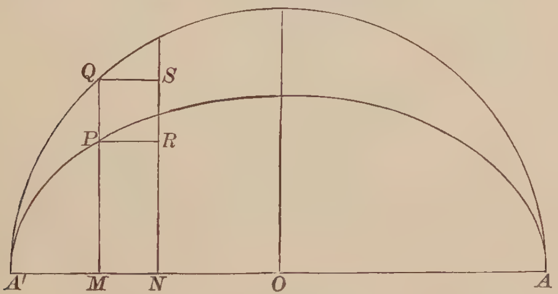


Fig. 58.

the ellipse in P and the auxiliary circle in Q. Through P, Q draw parallels to the axis of x, meeting the other ordinate in R, S, respectively. Then (§ 127)

$$\frac{\text{area of rectangle } MPRN}{\text{area of rectangle } MQSN} = \frac{MP}{MQ} = \frac{b}{a}.$$

A similar proportion holds true for every corresponding pair of rectangles.

Therefore, by the Theory of Proportion,

$$\frac{\text{sum of rectangles in ellipse}}{\text{sum of rectangles in circle}}=\frac{b}{a}.$$

This relation holds true however great the number of rectangles. The greater their number, the nearer does the sum of their areas approach the area of the elliptic quadrant in one case, and the circular quadrant in the other. In other words, these two quadrants are the *limits* of the sums of the two series of rectangles. Therefore, by the fundamental theorem of limits,

$$\frac{\text{area of elliptic quadrant}}{\text{area of circular quadrant}}=\frac{b}{a}.$$

Multiplying both terms of this ratio by 4,

$$\frac{\text{area of the ellipse}}{\text{area of the circle}}=\frac{b}{a}.$$

But the area of the circle $=\pi a^2$; therefore,

$$\textbf{area of the ellipse} = \pi ab. \qquad [31]$$

Exercise 32.

What are a, b, c, and e in the ellipse whose equation is:

1. $\frac{x^2}{25}+\frac{y^2}{16}=1$?
2. $x^2+2y^2=2$?
3. $3x^2+4y^2=12$?
4. $Ax^2+By^2=1$?

5. Find the latus rectum of the ellipse $3x^2+7y^2=18$.

6. Find the eccentricity of an ellipse if its latus rectum is equal to one-half its minor axis.

What is the equation of an ellipse if :

7. The axes are 12 and 8 ?

8. Major axis $= 26$, distance between foci $= 24$?

9. Sum of axes $= 54$, distance between foci $= 18$?

10. Latus rectum $= \frac{64}{5}$, eccentricity $= \frac{3}{5}$?

11. Minor axis $= 10$, distance from focus to vertex $= 1$?

12. The curve passes through $(1, 4)$ and $(-6, 1)$?

13. Major axis $= 20$, minor axis $=$ distance between foci ?

14. Sum of the focal radii of a point in the curve $= 3$ times the distance between the foci ?

15. Prove that the semi-minor axis is a mean proportional between the segments of the major axis made by one of the foci.

16. What is the ratio of the two axes if the centre and foci divide the major axis into four equal parts?

17. For what point of an ellipse is the abscissa equal to the ordinate ?

Find the intersections of the loci :

18. $3x^2 + 6y^2 = 11$ and $y = x + 1$.

19. $2x^2 + 3y^2 = 14$ and $y^2 = 4x$.

20. $x^2 + 7y^2 = 16$ and $x^2 + y^2 = 10$.

21. The ordinates of the circle $x^2 + y^2 = r^2$ are bisected ; find the locus of the points of bisection.

22. A straight line AB so moves that the points A and B always touch two fixed perpendicular straight lines. Show that any point P in AB describes an ellipse, and find its equation.

23. What is the locus of $Ax^2 + By^2 = C$ when C is zero? When is this locus imaginary?

24. Prove that the abscissas of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ are to the corresponding abscissas of the minor auxiliary circle, $x^2 + y^2 = b^2$, as $a : b$.

25. Construct an ellipse by the method of § 128.

26. Construct an ellipse, having given c and b.

27. Construct the axes of an ellipse, having given the foci and one point of the curve.

28. Construct the minor axis and foci, having given the major axis (in magnitude and position) and one point of the ellipse.

29. A square is inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Find the equations of the sides and the area of the square.

Tangents and Normals.

130. *To find the equations of a tangent and of a normal to an ellipse, having given the point of contact* (x_1, y_1).

Taking the equation of the ellipse,

$$b^2x^2 + a^2y^2 = a^2b^2,$$

and the equation of the straight line through (x_1, y_1) and (x_2, y_2),

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

and proceeding, as in § 64, we obtain as the equation of a secant through (x_1, y_1) and (x_2, y_2)

$$\frac{y - y_1}{x - x_1} = -\frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)}.$$

Now make $x_2 = x_1$, $y_2 = y_1$; then the chord becomes a tangent, and

$$\frac{y - y_1}{x - x_1} = -\frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)}$$

becomes
$$\frac{y - y_1}{x - x_1} = -\frac{b^2 x_1}{a^2 y_1},$$

which reduces to
$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1. \qquad [32]$$

From the equation above it appears that the value of the slope of the tangent, in terms of the coördinates of the point of contact, is

$$-\frac{b^2 x_1}{a^2 y_1}.$$

The normal is perpendicular to the tangent, and passes through (x_1, y_1); therefore, its equation is easily found (by the method of § 46) to be

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1). \qquad [33]$$

131. *To find the subtangent and subnormal.*

Making $y = 0$ in [32] and [33], and then solving the equations for x, we obtain:

Intercept of tangent on axis of $x = \frac{a^2}{x_1}$.

Intercept of normal on axis of $x = \frac{c^2}{a^2}x_1 = e^2 x_1$.

Whence, the values of the subtangent and the subnormal (defined as in § 63) are easily found to be as follows:

$$\textbf{Subtangent} = \frac{x_1^2 - a^2}{x_1}, \qquad [34]$$

$$\textbf{Subnormal} = -\frac{b^2}{a^2}x_1. \qquad [35]$$

132. *If tangents to ellipses having a common major axis are drawn at points having a common abscissa, they will meet on the axis of x.*

For in all these ellipses the values of a and x are constant, and therefore (by § 131) the tangents all cut the same intercept from the axis of x.

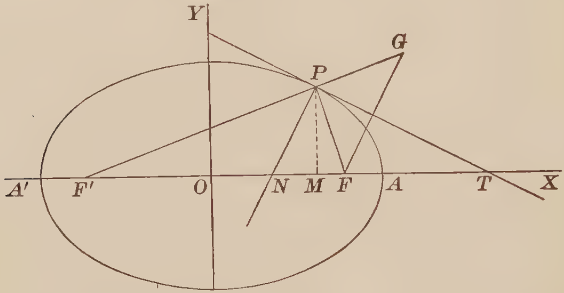


Fig. 59.

133. *The normal at any point of an ellipse bisects the angle formed by the focal radii.*

The values of the focal radii for the point P (Fig. 59) were found in § 119 to be

$$PF = a - ex_1, \quad PF' = a + ex_1.$$

If the normal through P meets the axis of x in N, $ON = e^2x_1$ (§ 131); and, therefore,

$$NF = c - e^2x_1 = ae - e^2x_1 = e(a - ex).$$
$$NF' = c + e^2x_1 = ae + e^2x_1 = e(a + ex).$$

Therefore, $NF : NF' = PF : PF'$,

or the normal divides the side FF' of the $\triangle\ PFF'$ into two parts proportional to the other two sides. Therefore (by Geometry), $\angle\ FPN = \angle\ F'PN$.

The tangent PT, being perpendicular to the normal, must bisect the angle FPG, formed by one focal radius with the other produced.

134. *To draw a tangent and a normal through a given point of an ellipse.*

I. Let P (Fig. 60) be the given point. Describe the auxiliary circle, draw the ordinate MP, produce it to meet the circle in Q, draw QT tangent to the circle and meeting the axis of x in T, and join PT: then PT is a tangent to the ellipse (§ 132). Draw $PN \perp$ to PT; PN is the normal at P.

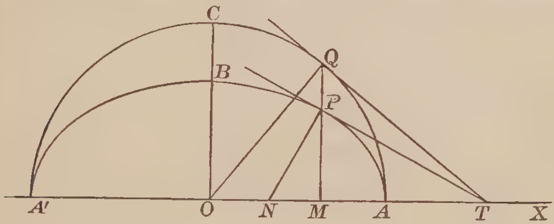


Fig. 60.

II. Draw the focal radii, and bisect the angles between them. The bisectors are the tangent and the normal at the point P (§ 133).

135. *To find the equation of a tangent to an ellipse in terms of its slope.*

This problem may be solved by finding under what condition the straight line

$$y = mx + c \tag{1}$$

will touch the ellipse $b^2x^2 + a^2y^2 = a^2b^2$. (2)

Eliminating y from (1) and (2), and then solving for x, we find two values of x:

$$x = \frac{-ma^2c \pm ab\sqrt{m^2a^2 + b^2 - c^2}}{m^2a^2 + b^2}.$$

These values will be equal if

$$m^2a^2 + b^2 - c^2 = 0, \text{ or } c = \pm\sqrt{m^2a^2 + b^2}.$$

If the two values of x are equal, the two values of y must also be equal, from equation (1).

Therefore, the two points in which the ellipse is cut by the line will coincide if $c = \pm\sqrt{m^2a^2+b^2}$.

Hence, the straight line

$$\boldsymbol{y = mx \pm \sqrt{m^2a^2+b^2}} \qquad [36]$$

will touch the ellipse for all values of m.

Since either sign may be given to the radical, it follows that *two* tangents having the same slope may be drawn to an ellipse.

136. *To find the locus of the intersection of two tangents to an ellipse which are perpendicular to each other.*

Let the equations of the tangents be

$$y = mx + \sqrt{m^2a^2+b^2}, \qquad (1)$$

$$y = m'x + \sqrt{m'^2a^2+b^2}. \qquad (2)$$

The condition to be satisfied is

$$mm' = -1, \text{ or } m' = -\frac{1}{m}.$$

If we substitute for m' in equation (2) its value in terms of m, the equations of the tangents may be written

$$y - mx = \sqrt{m^2a^2+b^2}, \qquad (3)$$

$$my + x = \sqrt{a^2+m^2b^2}. \qquad (4)$$

The coördinates, x and y, of the intersection of the tangents satisfy both (3) and (4); but before we can find the constant relation between them we must first eliminate the variable m.

This is most easily done by adding the squares of the two equations; the result is

$$(1+m^2)x^2 + (1+m^2)y^2 = (1+m^2)\ (a^2+b^2),$$

or

$$x^2 + y^2 = a^2 + b^2.$$

The required locus is therefore a circle. This circle is called the **Director Circle** of the ellipse.

Exercise 33.

1. What are the equations of the tangents and normals to the ellipse $2x^2+3y^2=35$ at the points whose abscissa $=2$?

2. What are the equations of the tangents and normals to the ellipse $4x^2+9y^2=36$ at the points whose abscissa $=-\frac{3}{2}$?

3. Find the equations of the tangent and the normal to the ellipse $x^2+4y^2=20$ at the point of contact $(2, 2)$. Also find the subtangent and the subnormal.

4. Show that the line $y=x+\sqrt{\frac{5}{6}}$ touches the ellipse $2x^2+3y^2=1$.

5. Required the condition which must be satisfied in order that the straight line $\frac{x}{m}+\frac{y}{n}=1$ may touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

6. In an ellipse the subtangent for the point $(3, \frac{12}{5})$ is $-\frac{16}{3}$, the eccentricity $=\frac{4}{5}$. What is the equation of the ellipse?

7. What is the equation of a tangent to the ellipse $9x^2+64y^2=576$ parallel to the line $2y=x$?

8. Find the equation of a tangent to the ellipse $3x^2+5y^2=15$ parallel to the line $4x-3y-1=0$.

9. In what points do the tangents that are equally inclined to the axes touch the ellipse $b^2x^2+a^2y^2=a^2b^2$?

10. Through what point of the ellipse $b^2x^2+a^2y^2=a^2b^2$ must a tangent and a normal be drawn in order that they may form, with the axis of x as base, an isosceles triangle?

11. Through a point of the ellipse $b^2x^2+a^2y^2=a^2b^2$, and the corresponding point of the auxiliary circle $x^2+y^2=a^2$, normals are drawn. What is the ratio of the subnormals?

12. For what points of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is the subtangent equal numerically to the abscissa of the point of contact?

13. Find the equations of tangents drawn from the point (3, 4) to the ellipse $16x^2 + 25y^2 = 400$.

14. What are the equations of the tangents drawn through the ends of the latera recta of the ellipse $4x^2 + 9y^2 = 36a^2$?

15. What is the distance from the centre of an ellipse to a tangent making the angle ϕ with the major axis?

16. What is the area of the triangle formed by the tangent in the last problem and the axes of coördinates?

17. From the point where the auxiliary circle cuts the minor axis produced tangents are drawn to the ellipse. Find the points of contact.

18. Prove that the tangents drawn through the ends of a chord through the centre are parallel.

19. Find the locus of the foot of a perpendicular dropped from the focus to a tangent.

Exercise 34. (Review.)

1. Given the ellipse $36x^2 + 100y^2 = 3600$. Find the equations and the lengths of focal radii drawn to the point $(8, \frac{18}{5})$.

2. Is the point (2, 1) within or without the ellipse $2x^2 + 3y^2 = 12$?

Find the eccentricity of an ellipse:

3. If the equation is $2x^2 + 3y^2 = 12$.

4. If the angle $FBF' = 90°$ (see Fig. 54).

5. Show that $4x^2 - 8x + 9y^2 - 36y + 4 = 0$, or $4(x-1)^2 + 9(y-2)^2 = 36$, is an ellipse whose centre is (1, 2), and semi-axes 3 and 2.

Find the equations of tangents to an ellipse:

6. If they make equal intercepts on the axes.

7. If they are parallel to BF (Fig. 54).

8. Which are parallel to the line $\frac{x}{a}+\frac{y}{b}=1$ (a and b being the semi-axes).

9. Find the equation of a tangent in terms of the eccentric angle ϕ of the point of contact.

Find the distance from the centre of an ellipse to:

10. A tangent through the point of contact (x_1, y_1).

11. A tangent making the angle ϕ with the axis of x.

12. In what ratio is the abscissa of a point divided by the normal at that point?

13. At the point (x_1, y_1) of an ellipse a normal is drawn. What is the product of the segments into which it divides the major axis?

14. Find the length of PN (Fig. 59).

15. Determine the value of the eccentric angle at the end of the latus rectum.

Prove that the semi-minor axis b of an ellipse is a mean proportional between:

16. The distances from the foci to a tangent.

17. A normal and the distance from the centre to the corresponding tangent.

Determine and describe the loci of the following points:

18. The middle point of the portion of a tangent contained between the tangents drawn through the vertices.

19. The middle point of a perpendicular dropped from a point of a circle $(x-a)^2+y^2=r^2$ to the axis of y.

20. The middle point of a chord of the ellipse $b^2x^2+a^2y^2=a^2b^2$ drawn through the positive end of the minor axis.

21. The vertex of a triangle whose base $2c$ and sum of the other sides $2s$ are given.

22. The vertex of a triangle, having given the base $2c$ and the product k of the tangents of the angles at the base.

23. The symmetrical point of the right-hand focus of an ellipse with respect to a tangent.

SUPPLEMENTARY PROPOSITIONS.

137. *Two distinct, two coincident, or no tangents can be drawn to an ellipse through any point* (h, k), *according as the point is without, on, or within the curve.*

Let the tangent $y=mx+\sqrt{m^2a^2+b^2}$ pass through the point (h, k); then

$$k=mh+\sqrt{m^2a^2+b^2},$$

or

$$(h^2-a^2)\,m^2-2hkm+k^2-b^2=0.$$

$$\therefore m=\frac{hk\pm\sqrt{b^2h^2+a^2k^2-a^2b^2}}{h^2-a^2}. \qquad (1)$$

Hence, there will be two distinct, two coincident, or no tangents through (h, k), according as $b^2h^2+a^2k^2-a^2b^2>$, $=$, or <0; that is, according as (h, k) is without, on, or within the ellipse.

138. *To find the equation of the chord of contact of the two tangents drawn from an external point* (h, k) *to the ellipse,*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Let the student prove, by a course of reasoning similar to that employed in §§ 71 and 104, that the required equation is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1.$$

139. *To find the equation of the polar of the pole* (h, k), *with regard to the ellipse.*

Let the student prove, by a course of reasoning similar to that employed in §§ 72 and 105, that the required equation is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1.$$

Cor. The tangent and chord of contact are particular cases of the polar. The proposition of § 74 holds true for poles and polars with regard to the ellipse.

140. *To draw a tangent to an ellipse from a given point* P *outside the curve.*

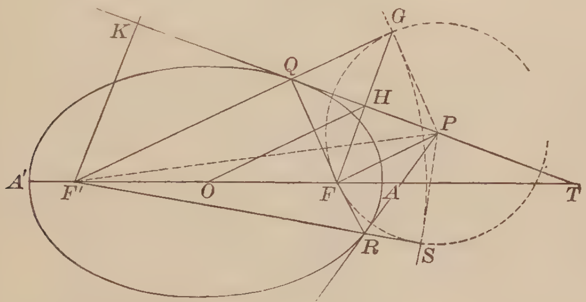


Fig. 61.

Suppose the problem solved, and let the tangent touch the ellipse at Q (Fig. 61). If $F'Q$ is produced to G, making $QG=QF$, then $\triangle\ FQG$ is isosceles; now $\angle\ FQP=\angle\ GQP$ (§ 133); therefore, PQ is perpendicular to FG at its middle point; therefore, P is equidistant from F and G. This reduces the problem to determining the point G.

Since $F'G=2a$, G lies on the circle with F' as centre and $2a$ as radius. And G also lies on the circle with P as centre and PF as radius. Hence the construction is obvious.

141. *To find the locus of the middle points of any system of parallel chords in the ellipse.*

Let any one of the parallel chords $y = mx + c$ meet the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2$$

in the points (x_1, y_1) and (x_2, y_2); then, by § 130,

$$m = -\frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)}. \tag{1}$$

If (x, y) is the middle point, $2x = x_1 + x_2$, $2y = y_1 + y_2$, and (1) becomes

$$m = -\frac{b^2x}{a^2y},$$

or

$$y = -\frac{b^2x}{a^2m}. \tag{2}$$

This relation holds true for the middle points of all the chords; therefore, it is the equation of the locus required.

From (2) we see that any straight line passing through the centre of an ellipse is a *diameter*.

142. Let m' denote the slope of the diameter of the chords whose slope is m; then from (2) of § 141

$$m' = -\frac{b^2}{ma^2}, \text{ or } \boldsymbol{mm' = -\frac{b^2}{a^2}}. \qquad [37]$$

Thus [37] is the equation of condition that the diameter $y = m'x$ bisects all chords parallel to the diameter $y = mx$; but [37] is evidently also the equation of condition that $y = mx$ bisects all chords parallel to $y = m'x$; hence,

If one diameter bisects all chords parallel to another, the second diameter bisects all chords parallel to the first.

Two such diameters are called **Conjugate Diameters.**

Cor. From [37] the slopes of two conjugate diameters must have opposite signs; hence, *two conjugate diameters of an ellipse lie on opposite sides of the minor axis.*

143. Let a straight line cutting the ellipse in P and Q move parallel to itself till P and Q coincide with the end of the diameter bisecting PQ; then the straight line becomes the tangent at the end of the diameter. Therefore,

The tangents at the extremities of any diameter are parallel to the chords of that diameter, and also to its conjugate diameter.

144. Let POP' and ROR' (Fig. 62) be two conjugate diameters meeting the ellipse in the points P (x_1, y_1) and R (x_2, y_2). The slope of the tangent through P is $-\dfrac{b^2x_1}{a^2y_1}$; hence, the equation of the diameter ROR', which is parallel to this tangent (§ 143), is

$$y = -\frac{b^2x_1}{a^2y_1}x, \text{ or } \frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 0. \tag{1}$$

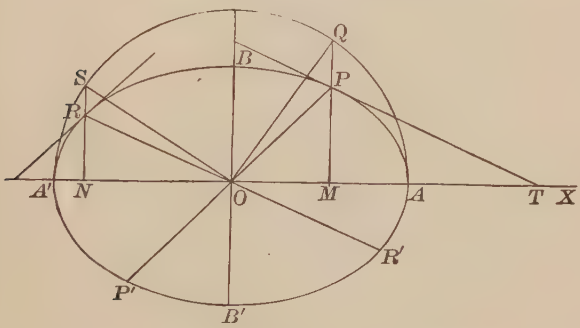


Fig. 62.

Now R (x_2, y_2) is on (1), and also on the ellipse; hence we have

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} = 0, \tag{2}$$

and

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1. \tag{3}$$

Solving (2) and (3) for x_2 and y_2, we obtain

$$x_2 = \mp\frac{a}{b}y_1, \; y_2 = \pm\frac{b}{a}x_1. \tag{4}$$

The upper signs give the coördinates of R, and the lower those of R' in terms of x_1 and y_1.

Equation (2) is the condition that must be satisfied by the coördinates of the extremities of every pair of conjugate diameters.

145. Denoting the semi-conjugate diameters OP and OR (Fig. 62) by a' and b', respectively, we have

$$a'^2 = x_1^2 + y_1^2 = x_1^2 + \frac{b^2}{a^2}(a^2 - x_1^2)$$

$$= b^2 + \frac{a^2 - b^2}{a^2} x_1^2 = b^2 + e^2 x_1^2, \tag{1}$$

and $$b'^2 = x_2^2 + y_2^2 = \frac{a^2}{b^2} y_1^2 + \frac{b^2}{a^2} x_1^2, \qquad (\S\ 144)$$

$$= a^2 - x_1^2 + \frac{b^2}{a^2} x_1^2 = a^2 - e^2 x_1^2. \tag{2}$$

Adding (1) and (2), we have

$$a'^2 + b'^2 = a^2 + b^2.$$

That is, *the sum of the squares of any pair of semi-conjugate diameters is equal to the sum of the squares of the semi-axes.*

Equations (1) and (2) express the lengths of the semi-conjugate diameters a' and b' in terms of a, b, and x_1 (the abscissa of the extremity of a').

146. Let the ordinates of the extremities P, R (Fig. 62) of two conjugate diameters meet the auxiliary circle in Q, S, respectively, join QO and SO, and denote $\angle\ QOX$ by ϕ, $\angle\ SOX$ by ϕ'. Then the values of the coördinates of P and R are (§ 127),

$$x_1 = a \cos \phi, \qquad x_2 = a \cos \phi',$$
$$y_1 = b \sin \phi, \qquad y_2 = b \sin \phi'.$$

Whence, by substitution in equation (2) of § 144, we obtain

$$\cos \phi \cos \phi' + \sin \phi \sin \phi' = 0.$$

Therefore, $\cos(\phi' - \phi) = 0$, or $\phi' - \phi = \frac{1}{2}\pi$.

That is, *the difference of the eccentric angles corresponding to the ends of two conjugate diameters is equal to a right angle.*

Cor. The angle POR (Fig. 62) is obtuse, since $QOS = \frac{1}{2}\pi$.

147. *To find the angle formed by two conjugate semi-diameters, whose lengths a', b' are given.*

Let the semi-diameters make the angles α, β, respectively, with the axis of x, and let θ denote the required angle. Then if (x_1, y_1) and (x_2, y_2) are the extremities of a' and b', respectively,

$$\sin\alpha = \frac{y_1}{a'},\quad \sin\beta = \frac{y_2}{b'} = \frac{bx_1}{ab'}.$$

$$\cos\alpha = \frac{x_1}{a'},\quad \cos\beta = \frac{x_2}{b'} = -\frac{ay_1}{bb'}.$$

$$\begin{aligned}\sin\theta &= \sin(\beta-\alpha)\\ &= \sin\beta\cos\alpha - \cos\beta\sin\alpha\\ &= \frac{b^2x_1^{\,2} + a^2y_1^{\,2}}{aba'b'} = \frac{a^2b^2}{aba'b'}\\ &= \frac{ab}{a'b'}.\end{aligned} \tag{1}$$

Cor. 1. Clearing (1) of fractions, we have

$$a'b'\sin\theta = ab,$$

which shows that the area of the parallelogram $HEKR$ is equal to the rectangle $LMQN$. ($\square\ CDRS = a'b'\sin\theta$.)

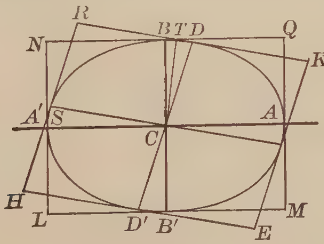


Fig. 63.

That is, *the parallelogram formed by tangents at the extremities of any pair of conjugate diameters is equal to the rectangle on the axes.*

Cor. 2. If CT (Fig. 63) is perpendicular to the tangent RK, then,

$$CT = CD \sin CDR = a' \sin \theta = \frac{ab}{b'}.$$

148. The lines joining any point of an ellipse to the ends of any diameter are called **Supplemental Chords.**

Let PQ, $P'Q$ be two supplemental chords (Fig. 64). Through the centre O draw OR parallel to $P'Q$, and meeting PQ in R; also OR' parallel to PQ, and meeting $P'Q$ in R'.

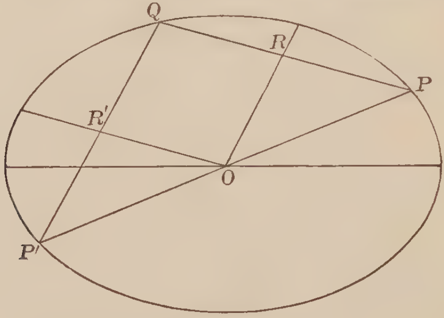


Fig. 64.

Since O is the middle point of PP', and OR is drawn parallel to $P'Q$, and OR' is drawn parallel to PQ, R and R' are the middle points of QP, QP', respectively. Therefore, OR will bisect all chords parallel to QP, and OR' will bisect all chords parallel to QP'. Hence, OR, OR' are conjugate diameters.

Therefore, *the diameters parallel to a pair of supplemental chords are conjugate diameters.*

Cor. 1. This principle affords the following easy method of drawing a pair of conjugate diameters which shall include a given angle.

On the transverse axis AA' describe a segment of a circle which shall include the given angle. Let the arc of this

segment cut the ellipse in Q and S; then the diameters parallel to QA and QA', or SA and SA', are conjugate and include the required angle.

COR. 2. If B is the upper vertex of the conjugate axis, the conjugate diameters parallel to BA and BA' will evidently be equal, and will lie on the diagonals of the rectangle on the axes of the ellipse.

149. *To find the equation of an ellipse referred to a pair of conjugate diameters as axes.*

Since each of two conjugate diameters of the ellipse bisects the chords parallel to the other, the curve is (obliquely) symmetrical with respect to each of the new axes; hence, as the required equation is of the second degree, it contains only the squares of x and y, and is of the form

$$Ax^2 + By^2 = C. \tag{1}$$

The intercepts of the curve on the new axes are the semi-conjugate diameters. Denoting them by a' and b', we have

$$A = \frac{C}{a'^2}, \qquad B = \frac{C}{b'^2}.$$

Substituting these values in (1), we obtain

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1, \tag{38}$$

which is the required equation in terms of the semi-conjugate diameters.

This equation has the same form as the equation referred to the axes of the curve; whence it follows that formulas derived from equation [30], by processes that do not presuppose the axes of coördinates to be rectangular, hold true when we employ as axes two conjugate diameters.

For example, the equation of a tangent at the point (x_1, y_1), referred to the semi-conjugate diameters a' and b', is

$$\frac{x_1 x}{a'^2} + \frac{y_1 y}{b'^2} = 1.$$

150. *To construct the polar of a focus.*

Since the polar of (h, k) is $\frac{hx}{a^2}+\frac{ky}{b^2}=1$, the polar of the focus $(ae, 0)$ is

$$aex=a^2, \text{ or } x=\frac{a}{e}.$$

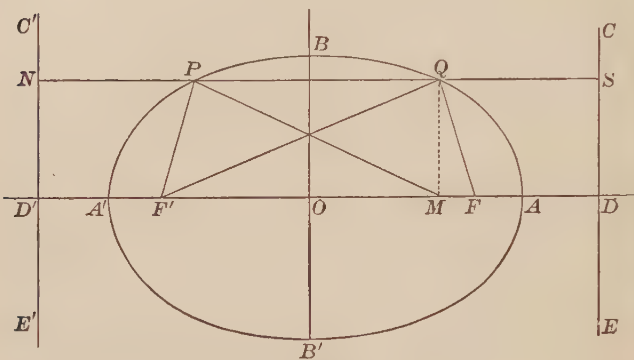


Fig. 65.

Hence, $$ae : a = a : x.$$

Therefore, if OD (Fig. 65) is taken so that

$$OF : OA = OA : OD,$$

and DC is drawn perpendicular to OD, DC will be the polar of the focus F.

The polar of a focus is called a **Directrix** of the ellipse. Hence, DC is the directrix corresponding to the focus F.

In like manner we may construct $E'C'$, or the *directrix* corresponding to the focus F'.

Cor. Let $Q(x, y)$ be any point on the ellipse; then,

$$QS = OD - OM = \frac{a}{e} - x = \frac{a-ex}{e} = \frac{FQ}{e}.$$

Hence, $e = FQ \div QS.$

That is, *the distances of any point on the ellipse from a focus, and the corresponding directrix, bear the constant ratio e.*

Whence, the ellipse is often defined as:

The locus of a point which moves so that its distances from a fixed point and a fixed straight line bear a constant ratio less than unity.

151. *To find the polar equation of the ellipse, the left-hand focus being taken as the pole.*

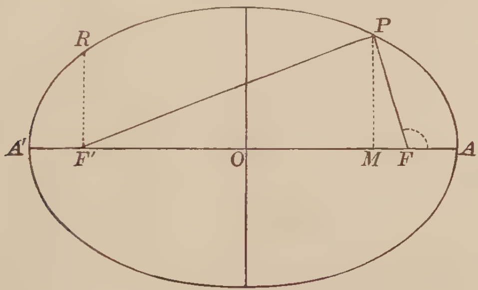


Fig. 66.

Let P be any point (ρ, θ) of the ellipse; then, from equation (8) of § 119, we have

$$\rho = a + ex. \tag{1}$$

Now $\quad x = OM = F'M - F'O = \rho \cos\theta - ae.$

Substituting this value of x in (1), we have

$$\rho = a + e\rho\cos\theta - ae^2.$$

Whence, $$\boldsymbol{\rho = \frac{a(1 - e^2)}{1 - e\cos\theta}}. \qquad [39]$$

Cor. Since $e < 1$, and $\cos\theta$ cannot exceed unity, ρ is always positive.

If $\quad \theta = 0,\ \rho = \dfrac{a(1 - e^2)}{1 - e} = a + ae = F'A.$

If $\quad \theta = \frac{1}{2}\pi,\ \rho = a(1 - e^2) = F'R =$ semi-latus rectum.

If $\theta = \pi, \quad \rho = \dfrac{a(1-e^2)}{1+e} = a - ae = F'A'.$

If $\theta = \frac{3}{2}\pi, \ \rho = a(1-e^2) =$ semi-latus rectum.

If $\theta = 2\pi, \ \rho = a + ae = F'A.$

While θ increases from zero to π, ρ decreases from $a + ae$ to $a - ae$; and while θ increases from π to 2π, ρ increases from $a - ae$ to $a + ae$.

If F is taken as the pole, the polar equation is

$$\rho = \frac{a(1-e^2)}{1 + e\cos\theta}.$$

Exercise 35.

1. Find the area of the ellipse $x^2 + 4y^2 = 16$.

2. Find the distances of the directrices from the centre in No. 1.

3. What is the equation of the polar of the point (5, 7) with respect to the ellipse $4x^2 + 9y^2 = 36$?

4. Prove that a focal chord is perpendicular to the line that joins its pole to the focus. In what line does the pole lie?

5. Find the pole of the line $Ax + By + C = 0$ with respect to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

6. Each of the two tangents that can be drawn to an ellipse from any point on its directrix subtends a right angle at the focus.

7. The two tangents that can be drawn to an ellipse from any external point subtend equal angles at the focus.

8. Find the slope m_1 of a diameter if the square of the diameter is (i) an arithmetic, (ii) a geometric, (iii) an harmonic mean between the squares of the axes.

9. Given the length $2l$ of a diameter, its inclination θ to the axis, and the eccentricity; find the major and minor axes.

10. Tangents at the extremities of any chord intersect on the diameter which bisects that chord.

11. Tangents are drawn from (3, 2) to the ellipse $x^2+4y^2=4$. Find the equation of the chord of contact, and of the line that joins (3, 2) to the middle point of the chord.

12. Find the area of the rectangle whose sides are the two segments into which a focal chord is divided by the focus.

13. What is the equation of a chord in the ellipse $13x^2+11y^2=143$ that passes through (1, 2) and is bisected by the diameter $3x-2y=0$?

14. In the ellipse $9x^2+36y^2=324$ find the equation of a chord passing through (4, 2) and bisected at this point.

15. Write the equations of diameters conjugate to the following lines:

$$x-y=0, \quad x+y=0, \quad ax=by, \quad ay=bx.$$

16. Show that the lines $2x-y=0$, $x+3y=0$ are conjugate diameters in the ellipse $2x^2+3y^2=4$.

17. Find the equation of a diameter parallel to the normal at the point (x_1, y_1), the semi-axes being a and b.

18. The rectangle of the focal perpendiculars upon any tangent is constant and equal to the square of the semi-minor axis.

19. The diagonals of the parallelogram in Fig. 63, § 147, are also conjugate diameters.

20. The angle between two semi-conjugate diameters is a maximum when they are equal.

21. The eccentric angles corresponding to equal semi-conjugate diameters are 45° and 135°.

22. The polar of a point in a diameter is parallel to the conjugate diameter.

23. Find the equations of equal conjugate diameters.

24. The length of a semi-diameter is l; find the equation of the conjugate diameter.

25. The angle between two equal conjugate diameters is $120°$; find the eccentricity of the ellipse.

26. Given a diameter, to construct the conjugate diameter.

27. To draw a tangent to a given ellipse parallel to a given straight line.

28. Given an ellipse, to find by construction the centre, foci, and axes.

29. Find the rectangular equation of the ellipse, taking the origin at the right-hand vertex.

30. Find the polar equation of an ellipse, taking as pole the right-hand focus.

31. Find the polar equation of the ellipse, taking the centre as pole.

32. If the centre of an ellipse is the point (4, 7), and the major and minor axes are 14 and 8, find its equation, the axes being supposed parallel to the axes of coördinates.

33. The equation of an ellipse, the origin being at the left-hand vertex, is $25x^2 + 81y^2 = 450x$; find the axes.

34. If the minor axis $= 12$, and the latus rectum $= 5$, what is the equation of the ellipse, the origin being taken at the left-hand vertex?

35. Find the eccentric angle ϕ corresponding to the diameter whose length is $2c$.

36. At the intersection of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and the circle $x^2 + y^2 = ab$ tangents are drawn to both curves. Find the angle between them.

37. How would you draw a normal to an ellipse from any point in the minor axis?

38. Find the equation of a chord that is bisected at the point (h, k).

39. Prove that the length of a line drawn from the centre to a tangent, and parallel to either focal radius of the point of contact, is equal to the semi-major axis.

40. A circle described on a focal radius will touch the auxiliary circle.

41. Find the locus of the intersection of tangents drawn through the ends of conjugate diameters of an ellipse.

42. Find the locus of the middle point of the chord joining the ends of two conjugate diameters.

43. Find the locus of the vertex of a triangle whose base is the line joining the foci, and whose other sides are parallel to two conjugated iameters.

44. Show that $4x^2+y^2+8x-2y+1=0$ represents an ellipse; find its centre and axes

45. If A and B have like signs, show that the locus of $Ax^2+By^2+Dx+Ey+F=0$ is in general an ellipse whose axes are parallel to the coördinate axes; and determine its semi-axes.

46. Find the locus of the centre of a circle that passes through the point $(0, 3)$ and touches internally the circle $x^2+y^2=25$.

CHAPTER VII.

THE HYPERBOLA.

Simple Properties of the Hyperbola.

152. The **Hyperbola** is the locus of a point the difference of whose distances from two fixed points is constant.

The fixed points are called the **Foci,** and a line joining any point of the curve to a focus is called a **Focal Radius.**

The constant difference is denoted by $2a$, and the distance between the foci by $2c$.

The fraction $\frac{c}{a}$ is called the **Eccentricity,** and is denoted by the letter e. Therefore, $c = ae$.

Since the difference of two sides of a triangle is always less than the third side, we must have in the hyperbola

$$2a < 2c, \text{ or } a < c, \text{ or } e > 1.$$

153. *To construct an hyperbola, having given the foci, and the constant difference 2a.*

I. *By Motion* (Fig. 67). Fasten one end of a ruler to one focus F' so that it can turn freely about F'. To the other end fasten a string. Make the length of the string less than that of the ruler by $2a$, and fasten the free end to the focus F. Press the string against the ruler by a pencil point P, and turn the ruler about F'.

The point P will describe one branch of an hyperbola. The other branch may be described in the same way by interchanging the fixed ends of the ruler and the string.

II. *By Points* (Fig. 68). Let F, F' be the foci; then
$$FF' = 2c.$$
Bisect FF' at O, and from O lay off $OA = OA' = a$.

Then
$$AA' = 2a, \quad FA = F'A'.$$
$$AF' - AF = AF' - A'F' = AA' = 2a.$$
$$A'F - A'F' = A'F - AF = AA' = 2a.$$
Therefore, A and A' are points of the curve.

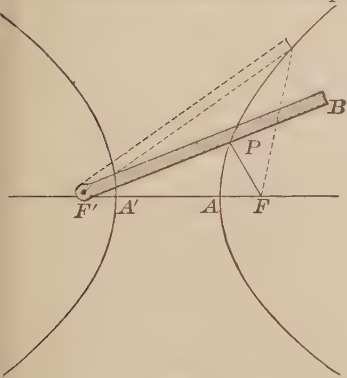


Fig. 67.

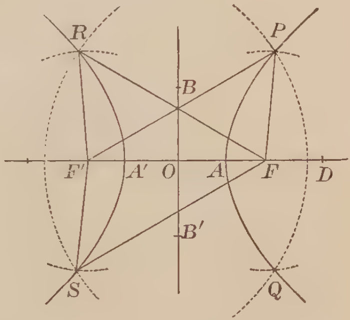


Fig. 68.

In FF' produced mark any point D; then describe two arcs, the first with F as centre and AD as radius, the second with F' as centre and $A'D$ as radius; the intersections P, Q of these arcs are points of the curve. By merely interchanging the radii, two more points R, S may be found.

Proceed in this way till a sufficient number of points has been obtained; then draw a smooth curve through them.

Through O draw $BB' \perp$ to FF'; since the difference of the distances of every point in the line BB' from the foci is 0, therefore the curve cannot cut the line BB'.

The locus evidently consists of two entirely distinct parts or *branches*, symmetrically placed with respect to the line BB', called the *right-hand* and the *left-hand* branches.

154. The point O, halfway between the foci, is the **Centre.**

The points A, A', where the line passing through the foci meets the curve, are called the **Vertices.**

The line AA' is the **Transverse Axis.**

The transverse axis is equal to the constant difference $2a$, and is bisected by the centre (§ 153).

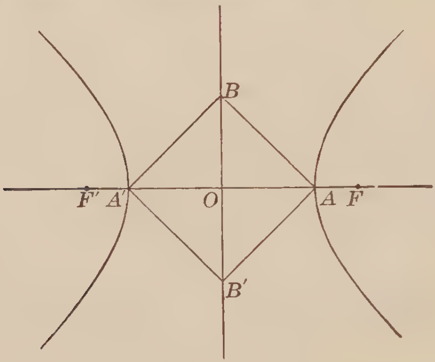


Fig. 69.

The line BB' passing through O perpendicular to AA' does not meet the curve (§ 153); but if B, B' are two points whose distances from the two vertices A, A' are each equal to c, then BB' is called the **Conjugate Axis,** and is denoted by $2b$.

Since $\triangle AOB = \triangle AOB'$, $OB = OB' = b$; that is, the conjugate axis is bisected by the centre.

In the triangle AOB, $OA = a$, $OB = b$, $AB = c$; hence,

$$c^2 = a^2 + b^2.$$

The chord passing through either focus perpendicular to the transverse axis is the **Latus Rectum,** or **Parameter.**

NOTE. Since a and b are equal to the *legs* of a right triangle, a may be greater than or less than b; hence the terms "*major*" and "*minor*" are not appropriate in the hyperbola.

155. By proceeding as in the case of the ellipse (§ 119), using $r' - r = \pm 2a$ instead of $r' + r = 2a$, and substituting b^2 for $c^2 - a^2$, we obtain as the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad [40]$$

Thus the equations of the ellipse and hyperbola differ only in the sign of b^2; that of the ellipse is changed into that of the hyperbola by substituting $-b^2$ for $+b^2$. Hence,

Any formula deduced from the equation of the ellipse is changed to the corresponding formula for the hyperbola by merely changing $+b^2$ to $-b^2$, or b to $b\sqrt{-1}$.

The lengths r, r' of the focal radii for any point (x, y) are

$$r = \pm(ex - a), \quad r' = \pm(ex + a),$$

in which the upper signs hold for the right-hand branch, and the lower for the left.

156. A discussion of equation [40] leads to the following conclusions:

(i) The curve cuts the axis of x at the two real points $(a, 0)$ and $(-a, 0)$.

(ii) The curve does not cut the axis of y. The imaginary intercepts are $\pm b\sqrt{-1}$.

(iii) No part of the curve lies between the straight lines $x = +a$ and $x = -a$.

(iv) Outside these lines the curve extends without limit both to the right and to the left.

(v) The greater the abscissa, the greater the ordinate.

(vi) The curve is symmetrical with respect to the axis of x.

(vii) The curve is symmetrical with respect to the axis of y.

(viii) Every chord that passes through the centre is bisected by the centre. This explains why the point halfway between the foci is called the centre.

157. An hyperbola whose transverse and conjugate axes are equal is called an **Equilateral Hyperbola.** Its equation is

$$x^2 - y^2 = a^2. \qquad [41]$$

The equilateral hyperbola bears to the general hyperbola the same relation that the auxiliary circle bears to the ellipse.

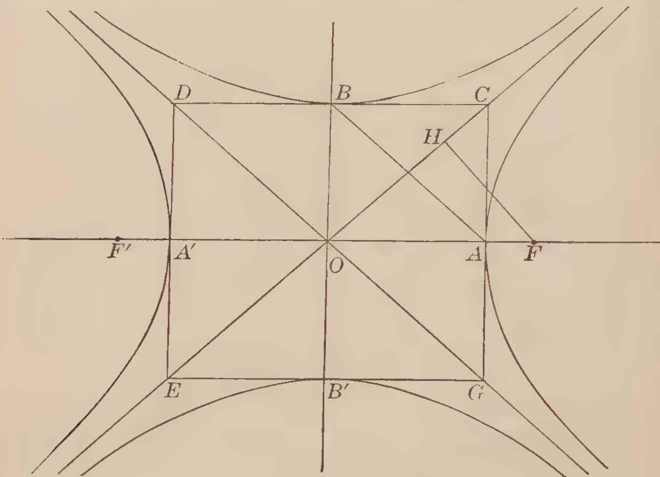


Fig. 70.

158. The hyperbola that has BB' for transverse axis, and AA' for conjugate axis, obviously holds the same relation to the axis of y that the hyperbola which has AA' for transverse axis and BB' for conjugate axis holds to the axis of x.

Therefore its equation is found by simply changing the signs of a^2 and b^2 in [40], and is

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ or } \frac{x^2}{a^2}-\frac{y^2}{b^2}=-1. \qquad (1)$$

The two hyperbolas are said to be **Conjugate.**

159. The straight line $y = mx$, passing through the centre of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, meets the curve in two points, the abscissas of which are

$$x_1 = \frac{+ab}{\sqrt{b^2 - m^2a^2}}, \quad x_2 = \frac{-ab}{\sqrt{b^2 - m^2a^2}}.$$

Hence the points will be *real*, *imaginary*, or *situated at infinity*, as $b^2 - m^2a^2$ is positive, negative, or zero; that is, as m^2 is *less than*, *greater than*, or *equal to* $\frac{b^2}{a^2}$.

The same line, $y = mx$, will meet the conjugate hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ in two points, whose abscissas are

$$x_1 = \frac{+ab,}{\sqrt{m^2a^2 - b^2}}, \quad x_2 = \frac{-ab}{\sqrt{m^2a^2 - b^2}}.$$

Hence these points will be *imaginary*, *real*, or *situated at infinity*, as m^2 is *less than*, *greater than*, or *equal to* $\frac{b^2}{a^2}$.

Whence,

If a straight line through the centre meets an hyperbola in imaginary points, it will meet the conjugate hyperbola in real points, and vice versa.

160. An **Asymptote** is a straight line that passes through finite points, and meets a curve in *two* points at infinity.

We see from § 159 that the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has two real asymptotes which pass through the centre of the curve, and which have for their equations $y = +\frac{b}{a}x$ and $y = -\frac{b}{a}x$; or,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0. \qquad [42]$$

Exercise 36.

What is the equation of an hyperbola, if :

1. Transverse axis $=16$, conjugate axis $=14$?

2. Conjugate axis $=12$, distance between foci $=13$?

3. Distance between foci $=$ twice the transverse axis?

4. Transverse axis $=8$, one point is $(10, 25)$?

5. Distance between foci $=2c$, eccentricity $=\sqrt{2}$?

6. Prove that the latus rectum of an hyperbola is equal to $\frac{2b^2}{a}$. Also $2a : 2b :: 2b :$ latus rectum.

7. The equation of an hyperbola is $9x^2-16y^2=144$; find the axes, distance between the foci, eccentricity, and latus rectum.

8. Write the equation of the hyperbola conjugate to the hyperbola $9x^2-16y^2=144$, and find its axes, distance between its foci, and its latus rectum.

9. If the vertex of an hyperbola bisects the distance from the centre to the focus, find the ratio of its axes.

10. Prove that the point (x, y) is *without*, *on*, or *within* the hyperbola, according as $\frac{x^2}{a^2}-\frac{y^2}{b^2}-1$ is *negative*, *zero*, or *positive*.

11. Find the eccentricity of an equilateral hyperbola.

12. Find the points that are common to the hyperbola $25x^2-9y^2=225$, and the straight line $25x+12y=45$.

13. The asymptotes of an hyperbola are the diagonals of the rectangle $CDEG$ (Fig. 70, p. 172).

14. Find the foci and the asymptotes of the hyperbola $16x^2-9y^2=144$.

15. The asymptotes of an equilateral hyperbola are perpendicular to each other. Hence the equilateral hyperbola is also called the *rectangular* hyperbola.

16. Two conjugate hyperbolas have the same asymptotes.

17. Find the length of the perpendicular dropped from the focus to an asymptote.

18. Prove that the squares of any two ordinates of an hyperbola are to each other as the products of the segments into which they divide the transverse axis externally.

TANGENTS AND NORMALS.

NOTE. The results stated in the following six sections may be established in the same way as the corresponding propositions in the ellipse, or the first five may be obtained by § 155.

161. The slope of the tangent at (x_1, y_1) is $\frac{b^2x_1}{a^2y_1}$, and its equation is

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1. \qquad [43]$$

162. The equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{a^2y_1}{b^2x_1}(x - x_1). \qquad [44]$$

163. The subtangent $= \frac{x_1{}^2 - a^2}{x_1}$, the subnormal $= \frac{b^2x_1}{a^2}$.

164. The straight line whose equation is

$$y = mx \pm \sqrt{m^2a^2 - b^2}$$

is a tangent for all values of m (§ 135).

165. The equation of the director circle of an hyperbola is $x^2 + y^2 = a^2 - b^2$ (§ 136).

166. The tangent and the normal at any point of an hyperbola bisect the angles formed by the focal radii of the point (§ 133).

Exercise 37.

1. Find the equations of the tangent and of the normal at the point (4, 4) of the hyperbola $16x^2 - 9y^2 = 112$. Also find the lengths of the subtangent and the subnormal.

2. Show that in an equilateral hyperbola the subnormal is equal to the abscissa of the point of contact.

3. The equations of the tangent and the normal at a point of an equilateral hyperbola are $5x - 4y = 9$, $4x + 5y = 40$. What is the equation of the hyperbola, and what are the coördinates of the point of contact?

4. For what points of an hyperbola is the subtangent equal to the subnormal?

5. To draw a tangent and a normal to an hyperbola at a given point of the curve.

6. If an ellipse and an hyperbola have the same foci, prove that the tangents to the two curves drawn at their points of intersection are perpendicular to each other.

7. Prove that the asymptotes of an hyperbola are the limiting positions of tangents to the infinite branches.

8. Prove that the length of a normal in an equilateral hyperbola is equal to the distance of the point of contact from the centre.

9. Find the distance from the origin to the tangent through the end of the latus rectum of the equilateral hyperbola $x^2 - y^2 = a^2$.

10. What condition must be satisfied in order that the straight line $\frac{x}{m} + \frac{y}{n} = 1$ may touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$?

11. When is the director circle of an hyperbola imaginary?

12. Find the locus of the foot of the perpendicular dropped from the focus of an hyperbola to a tangent.

Exercise 38. (Review.)

1. The ordinate through the focus of an hyperbola, produced, cuts the asymptotes in P and Q. Find PQ and the distances of P and Q from the centre.

2. In the hyperbola $9x^2-16y^2=144$ what are the focal radii of the points whose common abscissa is 8? What other points have equal focal radii?

3. What relation exists between the *sum* of the focal radii of a point of an hyperbola and the abscissa of the point?

4. Prove that in an equilateral hyperbola every ordinate is a mean proportional between the distances of its foot from the vertices of the curve. Hence, find a method of constructing an equilateral hyperbola when the axes are given.

5. In an equilateral hyperbola the distance of a point from the centre is a mean proportional between its focal radii.

6. In an equilateral hyperbola the bisectors of the angles formed by lines drawn from the vertices to any point of the curve are parallel to the asymptotes.

7. If e, e' are the eccentricities of two conjugate hyperbolas,

$$\frac{1}{e^2}+\frac{1}{e'^2}=1.$$

8. Through the positive vertex of an hyperbola a tangent is drawn. In what points does it cut the conjugate hyperbola?

9. The sum of the reciprocals of two focal chords perpendicular to each other is constant.

10. Through the foot of the ordinate of a point in an equilateral hyperbola a tangent is drawn to the circle described upon the transverse axis as diameter. What relation exists between the lengths of this tangent and the ordinate of the point?

11. In an equilateral hyperbola find the equations of tangents drawn from the positive end of the conjugate axis.

12. From what point in the conjugate axis of an hyperbola must tangents be drawn in order that they may be perpendicular to each other?

13. What condition must be satisfied that a square may be constructed whose sides shall be parallel to the axes of an hyperbola and whose vertices shall lie on the curve?

14. Find the equation of the chord of the hyperbola $16x^2-9y^2=144$ that is bisected at the point (12, 3).

15. Find the equation of the tangent to the hyperbola $16x^2-9y^2=144$ parallel to the line $y=4x-3$.

16. Find the product of the two perpendiculars let fall from any point of any hyperbola upon the asymptotes.

17. A chord of an hyperbola that touches the conjugate hyperbola is bisected at the point of contact.

SUPPLEMENTARY PROPOSITIONS.

NOTE. Many of the following propositions are closely analogous to propositions already established for the ellipse; hence the proofs are omitted, and references given to the chapter on the ellipse.

167. *Two distinct, two coincident, or no tangents can be drawn to an hyperbola through any point* (h, k), *according as the point is without, on, or within the curve* (§ 137).

168. *The equation of the chord of contact of the two tangents drawn from an external point* (h, k) *to the hyperbola*

$$\frac{x^2}{a^2}-\frac{y^2}{b_2}=1, \quad \text{is } \frac{hx}{a^2}-\frac{ky}{b^2}=1. \qquad (\S\ 138)$$

169. *The equation of the polar of the pole* (h, k) *with regard to the hyperbola is*

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1. \qquad (\S\ 139)$$

The tangent and chord of contact are particular cases of the polar, and the proposition of § 74 holds true for poles and polars with regard to the hyberbola.

170. *The equation of a diameter of an hyperbola is*

$$y = \frac{b^2}{a^2 m} x, \qquad (\S\ 141)$$

in which m is the slope of its chords.

171. If m' is the slope of the diameter bisecting the chords parallel to the diameter $y = mx$, then (§ 142)

$$mm' = \frac{b^2}{a^2}. \qquad [45]$$

Since m and m' are alike involved in [45], it follows that

If one diameter bisects all chords parallel to another, the second diameter will bisect all chords parallel to the first.

Two such diameters are called **Conjugate Diameters.**

172. From [45], the slopes of two conjugate diameters must agree in sign; hence,

Two conjugate diameters of an hyperbola lie on the same side of the conjugate axis, and their included angle is acute.

Also, if m in absolute magnitude is less than $\frac{b}{a}$, then m' must be greater than $\frac{b}{a}$. But the slope of the asymptotes is equal to $\pm \frac{b}{a}$. Therefore,

Two conjugate diameters lie on opposite sides of the asymptote in the same quadrant; and of two conjugate diameters, one meets the curve in real points and the other in imaginary points (§ 159).

173. The *length* of a diameter that meets the hyperbola in real points is the length of the chord between these points.

If a diameter meets the hyperbola in imaginary points, that is, does not meet it at all, it will meet the conjugate hyperbola in real points (§ 159); and its length is the length of the chord between these points. But from § 159 we know that if a diameter meet one of the hyperbolas in the imaginary point $(h\sqrt{-1},\ k\sqrt{-1})$, it will meet the other in the real point (h, k); hence, the length of the semi-diameter, which is $\sqrt{h^2+k^2}$, is known from the imaginary coördinates of intersection.

174. The equations of an hyperbola and its conjugate differ only in the signs of a^2 and b^2. But this interchange of signs does not affect the equation

$$mm' = \frac{b^2}{a^2}.$$

Therefore, *if two diameters are conjugate with respect to one of two conjugate hyperbolas, they will be conjugate with respect to the other.*

Thus, let POP' and QOQ' (Fig. 71) be two conjugate diameters. Then POP' bisects all chords parallel to QOQ' that lie *within* the branches of the original hyperbola and *between* the branches of the conjugate hyperbola; and QOQ' bisects all chords parallel to POP' that lie *within* the branches of the conjugate hyperbola and *between* the branches of the original hyperbola.

From the above theorem it follows immediately that

If a straight line meets each of two conjugate hyperbolas in two real points, the two portions of the line contained between the hyperbolas are equal (thus, $BD = B'D'$, Fig. 71).

175. *The tangent drawn through the end of a diameter is parallel to the conjugate diameter* (§ 143).

176. *Having given the end (x_1, y_1) of a diameter, to find the end (x_2, y_2) of the conjugate diameter.*

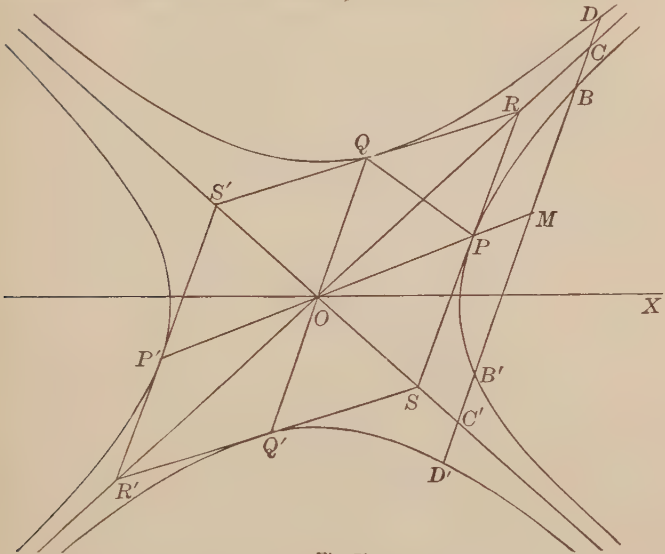


Fig. 71.

Let (x_1, y_1) be on the given hyperbola, then (x_2, y_2) is on the conjugate. The slope of the tangent at (x_1, y_1) is $\frac{b^2x_1}{a^2y_1}$; hence, the equation of the diameter conjugate to the diameter through (x_1, y_1) is

$$y = \frac{b^2x_1}{a^2y_1}x. \tag{1}$$

Now (x_2, y_2) is on the diameter (1) and also on the conjugate hyperbola; hence, we have

$$y_2 = \frac{b^2x_1}{a^2y_1}x_2, \quad \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} = -1. \tag{2}$$

Solving equations (2) for x_2 and y_2, we obtain

$$x_2 = \pm\frac{a}{b}y_1, \quad y_2 = \pm\frac{b}{a}x_1.$$

The positive signs belong to one end, and the negative signs to the other end, of the conjugate diameter.

177. *To find the equation of an hyperbola referred to any pair of conjugate diameters as axes of coördinates.*

From the symmetry of the curve with respect to each of the new axes, the required equation must be of the form

$$Ax^2 + By^2 = C.$$

Denoting the intercepts of the curve on the new axes by a' and $b'\sqrt{-1}$ (§ 172), we obtain

$$A = \frac{C}{a'^2}. \qquad B = -\frac{C}{b'^2}.$$

Whence,
$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1 \tag{1}$$

is the required equation, in which a' and b' are semi-conjugate diameters.

Since the form of equation (1) is the same as that of the equation referred to the axes of the curve, it follows that all formulas that have been obtained without assuming the axes of coördinates to be at right angles to each other hold good when the axes of coördinates are any two conjugate diameters. For example, the equation of the asymptotes of the hyperbola represented by equation (1) is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 0, \tag{2}$$

and the equation of the tangent is

$$\frac{x_1 x}{a'^2} - \frac{y_1 y}{b'^2} = 1. \tag{3}$$

178. *The tangents through the ends of two conjugate diameters meet in the asymptotes.*

The equations of these tangents referred to the conjugate diameters are

$$x = \pm a', \quad y = \pm b'.$$

Hence, their intersections are (a', b'), $(a', -b')$, $(-a', b')$, and $(-a', -b')$. But these points evidently lie upon the asymptotes, or the locus of (2) in § 177.

179. *If θ denotes the angle formed by two conjugate semi-diameters, and a' and b' their lengths, then* $\sin\theta = \dfrac{ab}{a'b'}$.

Substituting $b\sqrt{-1}$ for b, and $b'\sqrt{-1}$ for b' in equation (1) of § 147 and cancelling the imaginary factors, we obtain the above result.

Cor. 1. Since $4a'b'\sin\theta = 4ab$, the parallelogram $SRS'R'$ (Fig. 71) equals the rectangle on the axes of the curve.

Cor. 2. The length of the perpendicular from O upon the tangent $SPR = OP \sin OPS = a' \sin\theta = \dfrac{ab}{b'}$.

Cor. 3. From §§ 145, 155, 177, we have

$$a'^2 - b'^2 = a^2 - b^2.$$

180. *If a straight line cuts an hyperbola and its asymptotes, the portions of the line intercepted between the curve and its asymptotes are equal.*

Let CC' (Fig. 72) be the line meeting the asymptotes in C, C' and the curve in B, B', and let the equation of the line be

$$y = mx + c. \tag{1}$$

Let M be the middle point of the chord BB'; then (§ 170) the equation of the diameter through m is

$$y = \frac{b^2x}{a^2m}. \tag{2}$$

By combining equation (1) with the equations of the asymptotes, we obtain the coördinates of the points C and C'; taking the half-sum of these values, we get for the coördinates of the point halfway between C and C' the values

$$x = \frac{ma^2c}{b^2 - m^2a^2}, \quad y = \frac{b^2c}{b^2 - m^2a^2}.$$

These values satisfy equation (2); therefore, the point halfway between C and C' coincides with M; therefore, $MC = MC'$. And since $MB = MB'$, therefore, $BC = B'C'$.

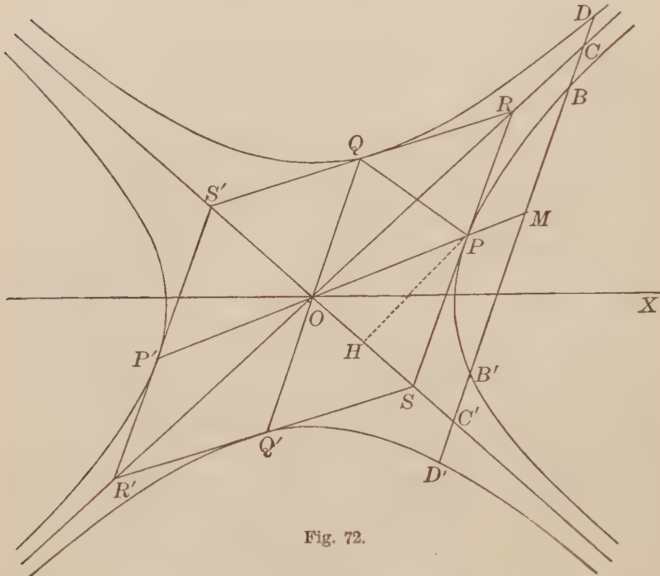


Fig. 72.

Cor. Let CC' be moved parallel to itself till it becomes a tangent at P, meeting the asymptotes in R, S; then the points B, B' coincide at P, and we have $PR = PS$. Hence,

The portion of a tangent intercepted by the asymptotes is bisected by the point of contact.

181. The following method of showing that an hyperbola has asymptotes, and finding their equations, is more general than the method given in §§ 159, 160.

The abscissas of the points where the straight line $y = mx + c$ meets an hyperbola are found by solving the equation

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1,$$

or
$$\frac{b^2 - m^2a^2}{a^2b^2}x^2 - \frac{2mc}{b^2}x = \frac{b^2+c^2}{b^2}. \qquad (1)$$

Now, from Algebra we know that as the coefficients of x^2 and x in (1) approach zero, both roots of (1) increase without limit. Hence, each root becomes infinity when

$$b^2 - m^2a^2 = 0, \text{ and } 2mc = 0,$$

or when
$$m = \pm\frac{b}{a}, \text{ and } c = 0.$$

Therefore, $y = \pm\frac{b}{a}x$ *are asymptotes to the hyperbola.*

If only $b^2 - m^2a^2 = 0$, then $m = \pm\frac{b}{a}$, the line is parallel to an asymptote, and one root of (1) is infinity, while the other is $-\frac{b^2+c^2}{2mc}$.

Hence, *a right line parallel to an asymptote meets the hyperbola in only one finite point.*

182. *To find the equation of an hyperbola referred to the asymptotes as axes of coördinates.*

Let the lines OB, OC (Fig. 73) be the asymptotes, A the vertex of the curve, and let the angle $AOC = \alpha$.

Let the coördinates of any point P of the curve be x, y when referred to the axes of the curve, and x', y' when referred to OB, OC as axes of coördinates.

Draw $PM \perp$ to OA, $PN \parallel$ to CO; then

$$x = ON\cos\alpha + NP\cos\alpha = (x'+y')\cos\alpha,$$
$$y = NP\sin\alpha - ON\sin\alpha = (y'-x')\sin\alpha.$$

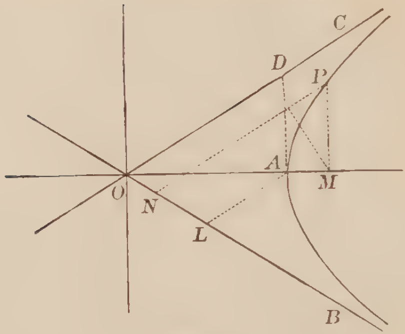


Fig. 73.

Hence, by substituting in [40], we obtain

$$\frac{(x'+y')^2\cos^2\alpha}{a^2}-\frac{(y'-x')^2\sin^2\alpha}{b^2}=1.$$

But $$\sin\alpha=\frac{AD}{OD}=\frac{b}{\sqrt{a^2+b^2}},$$

$$\cos\alpha=\frac{OA}{OD}=\frac{a}{\sqrt{a^2+b^2}}.$$

Substituting these values, and dropping accents, we have

$$4xy=a^2+b^2. \qquad [46]$$

Cor. 1. The equation of the conjugate hyperbola is

$$4xy=-(a^2+b^2).$$

Cor. 2. Sin $COB=\sin 2\alpha=2\sin\alpha\cos\alpha=\dfrac{2ab}{a^2+b^2}$.

If $a=b$, $\sin COB=1$; therefore, $COB=\frac{1}{2}\pi$.

Cor. 3. Let (x_1, y_1) denote P (Fig. 72), referred to the asymptotes; then

$$OS\times OR=2OH\times 2HP=4x_1y_1=a^2+b^2.$$

That is, *the product of the intercepts of a tangent on the asymptotes is equal to the sum of the squares of the semi-axes.*

Cor. 4. In Fig. 72, the area of the triangle ROS equals

$$\tfrac{1}{2}\, OS \times OR \sin ROS = \tfrac{1}{2}(a^2+b^2)\frac{2ab}{a^2+b^2} = ab.$$

That is, *the area of the triangle formed by any tangent and the asymptotes is equal to the product of the semi-axes.*

183. The polar of the focus $(ae, 0)$ is

$$x = \frac{a}{e} = \frac{a^2}{ae}.$$

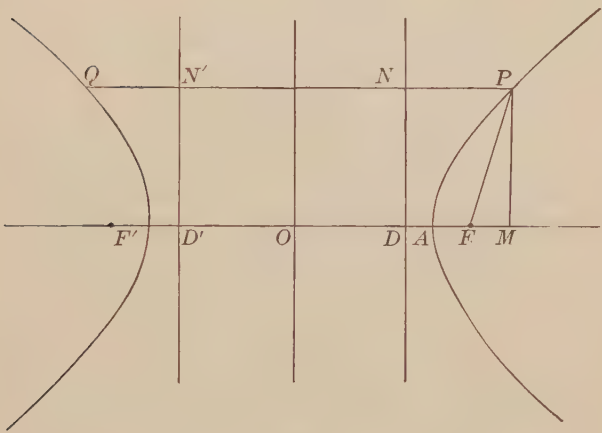


Fig. 74.

Hence, if OD is taken so that

$$OF : OA = OA : OD,$$

then DN perpendicular to OF is the polar of F, and is called a **Directrix** of the hyperbola. In like manner we may construct $D'N'$, or the directrix corresponding to the focus F'.

Cor. As in § 150 we may prove that

$$e = \frac{PF}{PN}.$$

Whence, the hyperbola may be defined as

The locus of a point whose distances from a fixed point and a fixed straight line bear a constant ratio greater than unity.

184. *To find the polar equation of an hyperbola, the left-hand focus being taken as pole.*

If x is reckoned from the centre, and we write

$$\rho = ex + a, \tag{1}$$

ρ will be positive or negative according as the point is on the right or left-hand branch.

Now $x = \rho \cos\theta - c = \rho\cos\theta - ae.$

Whence, by substitution and reduction,

$$\rho = \frac{a(e^2-1)}{e\cos\theta - 1}. \qquad [47]$$

From (1) we know that a point is on the right or left-hand branch, according as ρ in [47] is positive or negative; that is, according as $\cos\theta >$ or $< \frac{1}{e}$.

If $\theta = 0$, $\rho = ae + a = F'O + OA = F'A$ (Fig. 70).

If $e\cos\theta - 1 = 0$, or $\theta = \cos^{-1}\frac{1}{e}$, $\rho = \infty$, as it should, since in this case the radius vector is $\parallel$ to the asymptote.

If $\theta = \frac{1}{2}\pi$, $\rho = -a(e^2-1) = -$ semi-latus rectum.

If $\theta = \pi$, $\rho = a - ae = -F'A'$.

Exercise 39.

1. What is the polar of the point $(-9, 7)$ with respect to the hyperbola $7x^2 - 12y^2 = 112$?

2. Find the equations of the directrices of an hyperbola.

3. Find the angle formed by a focal chord and the line that joins its pole to the focus.

4. Find the pole of the line $Ax+By+C=0$ with respect to an hyperbola.

5. Find the polar of the right-hand vertex of an hyperbola with respect to the conjugate hyperbola.

6. Find the distance from the centre of an hyperbola to the point where the directrix cuts the asymptote.

7. If (x_1, y_1) and (x_2, y_2) are the ends of two conjugate diameters, then

$$\frac{x_1x_2}{a^2}-\frac{y_1y_2}{b^2}=0.$$

8. The equation of a diameter in the hyperbola $25x^2-16y^2=400$ is $3y=x$. Find the equation of the conjugate diameter.

9. In the hyperbola $49x^2-4y^2=196$, find the equation of that chord which is bisected at the point (5, 3).

10. Find the length of the semi-diameter conjugate to the diameter $y=3x$ in the hyperbola $9x^2-4y^2=36$.

11. Two tangents to an hyperbola at right angles intersect on the circle

$$x^2+y^2=a^2-b^2.$$

12. Tangents at the extremities of any chord of an hyperbola intersect on the diameter which bisects that chord.

13. Prove that PQ (Fig. 71) is parallel to one asymptote and bisected by the other.

14. An asymptote is its own conjugate diameter.

15. The conjugate diameters of an equilateral hyperbola are equal.

16. Having given two conjugate diameters in length and position, to find by construction the asymptotes and the axes.

17. To draw a tangent to an hyperbola from a given point.

18. Find the equation of a tangent at any point (x_1, y_1) of the hyperbola $4xy = a^2 + b^2$.

19. Find the equation of an hyperbola, taking as the axis of y

(i) the tangent through the left-hand vertex;
(ii) the tangent through the right-hand vertex.

20. Find the polar equation of an hyperbola, taking the right-hand focus as pole.

21. Find the polar equation of an hyperbola, taking the centre as pole.

22. To find the centre of a given hyperbola.

23. The distance from a fixed point to a fixed straight line is 10. Find the locus of a point which moves so that its distance from the fixed point is always twice its distance from the fixed line.

24. Show that the locus of $x^2 - 4y^2 - 2x - 16y - 19 = 0$ is an hyperbola; find its centre and axes.

25. If A and B have unlike signs, prove that the locus of $Ax^2 + By^2 + Dx + Ey + F = 0$ is in general an hyperbola whose axes are parallel to the coördinate axes; and determine its semi-axes.

26. Through the point $(-4, 7)$ a straight line is drawn to meet the axes of coördinates, and then revolved about this point. Find the locus of the point midway between the axes.

27. A straight line has its ends in two fixed perpendicular lines, and forms with them a triangle of constant area a^2. Find the locus of the middle point of the line.

28. The base a of a triangle is fixed in length and position, and the vertex so moves that one of the base angles is always double the other. Find the locus of the vertex.

CHAPTER VIII.

LOCI OF THE SECOND ORDER.

185. The loci represented by equations of the second degree that are not of the first order are called **Loci of the Second Order.**

In the preceding chapters we have seen that the circle, parabola, ellipse, and hyperbola are loci of the second order. We will now inquire whether there are other loci of the second order besides the four curves just named; in other words, we will determine what loci may be represented by equations of the second degree.

For this purpose we shall write the general equation of the second degree in the form

$$Ax^2+By^2+Cxy+Dx+Ey+F=0, \tag{1}$$

and shall assume that the axes of coördinates are rectangular. This assumption will in nowise diminish the generality of our conclusions; for if the axes were oblique, we could refer the equation to rectangular axes, and this change would not alter the degree of the equation or the nature of the locus which it represents (§ 91).

186. *To find the condition that the general equation of the second degree may represent two loci of the first order.*

To do this let us solve (1) with respect to one of the variables. Choosing y for this purpose, we obtain

$$y=-\frac{Cx+E}{2B}\pm\frac{1}{2B}\sqrt{Lx^2+Mx+N}, \tag{2}$$

where $L=C^2-4AB,\ M=2(CE-2BD),\ N=E^2-4BF.$

If $Lx^2 + Mx + N$ is a perfect square, then the locus of (2), or (1), will be two loci of the first order.

Now, from Algebra, we know that the condition that $Lx^2 + Mx + N$ should be a perfect square is

$$M^2 - 4LN = 0;$$

or, substituting the values of L, M, and N, we have

$$(CE - 2BD)^2 - (C^2 - 4AB)(E^2 - 4BF) = 0,$$

or $$F(C^2 - 4AB) + AE^2 + BD^2 - CDE = 0. \qquad (3)$$

The quantity on the left-hand side of equation (3) is usually denoted by Δ, and is called the **Discriminant** of equation (1).

This same result was obtained by a more general method in § 57; hence,

Whenever $\Delta = 0$, *equation* (1) *represents two loci of the first order.* These loci may be readily determined by resolving (1) into two simple equations in x and y.

CENTRAL CURVES. Σ NOT ZERO.

187. A centre of a curve is a point that bisects every chord passing through it. Loci are classified as **Central** and **Non-Central,** according as they have or have not a definite centre. The circle, ellipse, and hyberbola belong to the first class, the parabola to the second.

188. *To find the equation of the central loci represented by equation* (1) *referred to their centre.*

To do this let us change the origin to the point (h, k), and then so choose the values of h and k that the terms involving the first powers of x and y will vanish. Making the change by substituting in (1) $x + h$ for x, and $y + k$ for y, we find that the coefficients A, B, and C remain unaltered, and we may write the transformed equation

$$Ax^2 + By^2 + Cxy + D'x + E'y = R, \qquad (4)$$

where $$D' = 2Ah + Ck + D,$$
$$E' = 2Bk + Ch + E,$$
$$R = -[Ah^2 + Bk^2 + Chk + Dh + Ek + F].$$

The values of h and k that will make D' and E' vanish are evidently found by solving the equations

$$2Ah + Ck + D = 0,$$
$$2Bk + Ch + E = 0,$$

and are $$h = \frac{CE - 2BD}{4AB - C^2}, \quad k = \frac{CD - 2AE}{4AB - C^2}.$$

If $4AB - C^2$, denoted by Σ, is not zero, these values of h and k are finite and single, and equation (4) may be written

$$Ax^2 + By^2 + Cxy = R. \tag{5}$$

From the form of (5) we see that if (x, y) is a point in its locus, so also is $(-x, -y)$; that is, the new origin (h, k) is the centre of the locus. Hence,

When Σ is not zero, equation (1) *can be reduced to the form of* (5), *and represents central curves.*

When, however, $\Sigma = 0$, the values of h and k become infinite or indeterminate, and the locus of (1) has no definite centre. Hence,

When $\Sigma = 0$, (1) *cannot be reduced to the form of* (5), *and represents non-central curves.*

The value of R can be reduced to the following useful form, which shows also that R and Δ vanish together.

$$\begin{aligned}
R &= -[Ah^2 + Bk^2 + Chk + Dh + Ek + F] \\
&= -\tfrac{1}{2}[(2Ah + Ck + D)h \\
&\qquad + (2Bk + Ch + E)k + Dh + Ek + 2F] \\
&= -\tfrac{1}{2}(D'h + E'k + Dh + Ek + 2F) \\
&= -\tfrac{1}{2}(Dh + Ek + 2F) \\
&= -\tfrac{1}{2}\frac{2BD^2 - CDE + 2AE^2 - CDE + 2F(C^2 - 4AB)}{C^2 - 4AB} \\
&= \frac{\Delta}{\Sigma}.
\end{aligned}$$

189. *To reduce* (5) *to a known form by causing the term in xy to disappear.*

For this purpose we change the direction of the axes through an angle θ, keeping the origin unaltered, and then determine the value of θ by putting the new term that involves xy equal to zero.

The change is made by substituting for x and y, in equation (5), the respective values (§ 86),

$$x \cos\theta - y \sin\theta,$$
$$x \sin\theta + y \cos\theta;$$

and equation (5) now becomes

$$Px^2 + Qy^2 + C'xy = R,$$

where
$$P = A\cos^2\theta + B\sin^2\theta + C\sin\theta\cos\theta, \tag{6}$$
$$Q = A\sin^2\theta + B\cos^2\theta - C\sin\theta\cos\theta, \tag{7}$$
$$C' = 2(B - A)\sin\theta\cos\theta + C(\cos^2\theta - \sin^2\theta). \tag{8}$$

Putting $C' = 0$, we obtain, by Trigonometry,

$$(A - B)\sin 2\theta - C\cos 2\theta = 0, \tag{9}$$

or
$$\tan 2\theta = \frac{C}{A - B}. \tag{10}$$

Since any real number, positive or negative, is the tangent of some angle between zero and π, equation (10) is satisfied by some value of θ between zero and $\frac{1}{2}\pi$. In what follows we shall use the simplest root of (10).

By this transformation, equation (5) is reduced to the form

$$Px^2 + Qy^2 = R, \tag{11}$$

of which the discussion will be found in the next section.

Cor. 1. The values of P and Q in terms of A, B, C may be found as follows:

From (6) and (7), by addition and subtraction,

$$P + Q = A + B, \tag{12}$$
$$P - Q = (A - B)\cos 2\theta + C\sin 2\theta. \tag{13}$$

Equation (9) may be written

$$0 = (A - B)\sin 2\theta - C\cos 2\theta. \qquad (14)$$

Adding the squares of (13) and (14), we have

$$(P - Q)^2 = (A - B)^2 + C^2, \qquad (15)$$

or

$$P - Q = \pm\sqrt{(A - B)^2 + C^2}. \qquad (16)$$

Whence, from (12) and (16),

$$P = \tfrac{1}{2}[A + B \pm \sqrt{(A - B)^2 + C^2}], \qquad (17)$$

$$Q = \tfrac{1}{2}[A + B \mp \sqrt{(A - B)^2 + C^2}]. \qquad (18)$$

These values of P and Q are evidently always real.

Cor. 2. By squaring (12) and subtracting (15), we obtain

$$4PQ = 4AB - C^2 = \Sigma. \qquad (19)$$

Hence, *P and Q have like or unlike signs, according as Σ is positive or negative.*

Cor. 3. In applying formulas (17) and (18), the question arises which sign before the radical should be used. If in (13) we substitute for $\cos 2\theta$, its value obtained from (14), we have

$$P - Q = \frac{[(A - B)^2 + C^2]\sin 2\theta}{C}.$$

Since the numerator of the fraction is always positive, $P - Q$ must have the same sign as C; that is, the upper or lower sign in (16) must be taken according as C is positive or negative.

Hence, *the upper or lower signs in* (17) *and* (18) *are to be taken according as C is positive or negative.*

190. The nature of the locus represented by equation (11) depends upon the signs of P, Q, and R. There are two groups of cases, according as Σ is positive or negative, and three cases in each group.

Group 1. Σ *Positive.*

In this group, P and Q must, by (19), agree in sign.

CASE 1. If R agrees in sign with P and Q, then, by § 124, the locus is an ellipse whose semi-axes are $\sqrt{\frac{R}{P}}$ and $\sqrt{\frac{R}{Q}}$.

If $P=Q$, the locus is a circle.

CASE 2. If R differs from P and Q in sign, no real values of x and y will satisfy (11), so that no real locus exists.

CASE 3. If $R=0$, the locus is the single point $(0, 0)$.

Group 2. Σ *Negative.*

In this group, P and Q, by (19), must have unlike signs.

CASE 1. If R agrees in sign with P, we may, by division (and by changing the signs of all the terms if necessary), put equation (11) into the form of equation [40], page 171. Therefore, the locus is an *hyperbola*, with its transverse axis on the axis of x, and having for semi-axes

$$a=\sqrt{\frac{R}{P}},\quad b=\sqrt{\frac{R}{-Q}}.$$

CASE 2. If R agrees in sign with Q, we may, by division (and by change of signs if necessary), put equation (11) into the form of equation (1), page 172. Therefore, the locus is an *hyperbola*, with its transverse axis on the axis of y.

CASE 3. If $R=0$, the locus consists of *two straight lines*, intersecting at the origin, and having for their equations

$$y=\pm\sqrt{\frac{P}{-Q}}\,x.$$

NON-CENTRAL CURVES. $\Sigma=0$.

191. *To determine the locus of* (1) *when* Δ *and* Σ *are both zero.*

If $\Delta = \Sigma = 0$, then from the first form of Δ in § 186 we must have

$$CE - 2BD = 0. \tag{20}$$

Hence, $L = M = 0$, (§ 186)

and (2) becomes

$$2By + Cx + E \mp \sqrt{E^2 - 4BF} = 0, \tag{21}$$

which represents two parallel straight lines, two coincident straight lines, or no locus, according as $E^2 - 4BF >, =,$ or < 0.

When $4AB - C^2 = 0$, if C is not zero, neither A nor B can be zero; if $C = 0$, either A or B must be zero, but both cannot be, since if $A = B = C = 0$, (1) is no longer of the second degree.

When $C = B = 0$, by solving (1) for x, and introducing the above condition, we should obtain, instead of (21) its corresponding equation,

$$2Ax + Cy + D \mp \sqrt{D^2 - 4AF} = 0, \tag{22}$$

whose locus is also two parallel straight lines. Hence,

When Δ and Σ are both zero, equation (1) *represents two parallel straight lines, two coincident straight lines, or no locus.*

Cor. Eliminating B between $\Sigma = 0$ and (20), we obtain

$$CD - 2AE = 0. \tag{23}$$

In like manner (20) follows from $\Sigma = 0$ and (23).

From these results, and the values of h and k, we learn that

(i) When $\Delta = \Sigma = 0$, h and k are both indeterminate, and conversely.

(ii) The values of h and k are indeterminate together.

192. *To determine the locus of* (1) *when Σ is zero and Δ is not zero.*

We simplify (1) by first making the term in xy disappear by proceeding exactly as in § 189; that is, by turning the axes

through an angle θ, the value of which is determined by the equation

$$\tan 2\theta = \frac{C}{A-B}. \tag{24}$$

If P, Q, U, V represent the new coefficients of x^2, y^2, x, y, respectively, P and Q will have values identical with those of P and Q given in § 189, and

$$U = \quad D\cos\theta + E\sin\theta, \tag{25}$$
$$V = -D\sin\theta + E\cos\theta. \tag{26}$$

Since $C^2 = 4AB$, from (17) and (18) we have, when C is positive,

$$P = A + B,\ Q = 0.$$

When C is negative,

$$P = 0,\ Q = A + B,$$

and (1) assumes the form

$$Qy^2 + Ux + Vy + F = 0. \tag{27}$$

To further simplify, we divide by Q, and obtain

$$y^2 + \frac{U}{Q}x + \frac{V}{Q}y + \frac{F}{Q} = 0,$$

or

$$y^2 + \frac{V}{Q}y + \frac{V^2}{4Q^2} + \frac{U}{Q}\left(x + \frac{F}{U} - \frac{V^2}{4QU}\right) = 0,$$

or

$$\left(y + \frac{V}{2Q}\right)^2 = -\frac{U}{Q}\left(x + \frac{4QF - V^2}{4UQ}\right). \tag{28}$$

If we now take as a new origin the point

$$\left(-\frac{4QF - V^2}{4UQ},\ -\frac{V}{2Q}\right),$$

equation (28) becomes

$$y^2 = -\frac{U}{Q}x,$$

which represents a parabola whose axis coincides with the axis of x, and which is situated on the *positive* or the *negative* side of the new origin, according as U and Q are *unlike* or *like* in sign (§ 94).

The vertex of the parabola is the new origin, and the parameter is equal to the coefficient of x in the equation of the curve.

This last transformation is possible except when $U=0$; but, when $U=0$, (27) evidently represents two parallel straight lines.

Suppose that C is positive. Then the general equation becomes

$$Px^2+Ux+Vy+F=0. \tag{29}$$

And this, by changing the origin to the point

$$\left(-\frac{4PF-U^2}{4VP},\ -\frac{U}{2P}\right)$$

becomes $$x^2=-\frac{V}{P}y.$$

This represents a *parabola* having the axis of y for its axis, and placed on the *positive* or the *negative* side of the new origin, according as V and P are *unlike* or *like* in sign.

It should be noted that the value of P or Q, when not zero, is $A+B$. The values of U and V* can be found from (25) and (26).

---

* We may obtain the values of U and V in terms of the original coefficients, as follows:

From (24) we find, by Trigonometry,

$$\tan\theta=\frac{-(A-B)\pm\sqrt{(A-B)^2+C^2}}{C}.$$

Introducing the condition $4AB=C^2$, we obtain

$$\tan\theta=-\frac{2A}{C},\quad \text{if } C \text{ is negative};$$

$$=\frac{2B}{C},\quad \text{if } C \text{ is positive};$$

whence, if C is negative,

$$\sin\theta=\frac{2A}{\sqrt{4A^2+C^2}},\quad \cos\theta=\frac{-C}{\sqrt{4A^2+C^2}}.$$

And if C is positive,

If $C = A = 0$, the given equation is of the form of (27), its locus is a parabola and can be found as that of (27). If $C = B = 0$, the given equation is of the form of (29), and its locus is a parabola.

193. The main results of the investigation are given in the following Table:

Loci Represented by the General Equation of the Second Degree.

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0.$$

CLASS.	CONDITIONS.	LOCUS.
I. Loci having a centre.	Σ positive, Δ not zero.	Ellipse, or no locus.
	Σ positive, $\Delta = 0$.	Point.
	Σ negative, Δ not zero.	Hyperbola.
	Σ negative, $\Delta = 0$.	Two intersecting straight lines.
II. Loci not having a centre.	$\Sigma = 0$, Δ not zero.	Parabola.
	$\Sigma = 0$, $\Delta = 0$.	Two parallel straight lines, One straight line, Or no locus.

Thus it appears that there are no loci of the second order besides those whose properties have been studied in the preceding chapters.

$$\sin\theta = \frac{2B}{\sqrt{4B^2 + C^2}}, \quad \cos\theta = \frac{C}{\sqrt{4B^2 + C^2}}.$$

By substitution, we obtain from (25) and (26),

if C is negative, $\quad U = \dfrac{2AE - CD}{\sqrt{4A^2 + C^2}}, \quad V = -\dfrac{2AD + CE}{\sqrt{4A^2 + C^2}};$ (30)

if C is positive, $\quad U = \dfrac{2BE + CD}{\sqrt{4B^2 + C^2}}, \quad V = \dfrac{CE - 2BD}{\sqrt{4B^2 + C^2}}.$ (31)

194. EXAMPLES. 1. Determine the nature of the locus

$$5x^2+5y^2+2xy-12x-12y=0, \tag{1}$$

transform the equation and construct it.

Here $A=5, B=5, C=2, D=-12, E=-12, F=0.$

Whence, $\Sigma=96$, $\Delta=1152$.

Hence, the equation represents an ellipse or no real locus.

$$R=\Delta\div\Sigma=12,\ h=1,\ k=1.$$

Therefore, the equation of the locus referred to new parallel axes through the centre (1, 1) is (§ 188)

$$5x^2+2xy+5y^2=12. \tag{2}$$

To cause the term in xy to disappear, we have

$$\tan 2\theta=\frac{C}{A-B}=\frac{2}{0}=\infty.$$

Whence, $2\theta=90°$, or $\theta=45°$.

$$P=\tfrac{1}{2}[A+B+\sqrt{(A-B)^2+C^2}]=6.$$
$$Q=\tfrac{1}{2}[A+B-\sqrt{(A-B)^2+C^2}]=4.$$

(We use the upper signs in the values of P and Q by § 189.)

Hence, by § 189 the equation of the ellipse referred to its own axes is

$$6x^2+4y^2=12, \quad \text{or } \frac{x^2}{2}+\frac{y^2}{3}=1. \tag{3}$$

Whence, $a=\sqrt{3}$, $b=\sqrt{2}$, and a lies on the axis of y.

To construct the equation, draw the axes O_1X_1, O_1Y_1 (Fig. 75); locate the centre (1, 1). Through this point O_2 draw the second set of axes, O_2X_2, O_2Y_2. Through O_2 draw the third set of axes O_2X_3, O_2Y_3, making $X_2O_2X_3$ equal to 45°.

Lay off $O_2A'=O_2A=\sqrt{3}$, and $O_2O_1=O_2B=\sqrt{2}$. The ellipse having BO_1 and AA' as axes will be the required locus.

The equation of the locus referred to the axes O_1X_1, O_1Y_1 is (1); its equation referred to O_2X_2, O_2Y_2 is (2); and its equation referred to O_2X_3, O_2Y_3 is (3). In constructing the locus it is not necessary to draw the second set of axes O_2X_2, O_2Y_2.

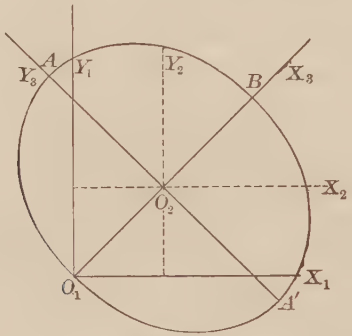


Fig. 75.

2. Determine the nature of the locus

$$x^2+y^2-5xy+8x-20y+15=0, \tag{1}$$

transform the equation and construct it.

Here $\quad \Sigma=-21, \quad \Delta=-21.$

Therefore, the locus is an hyperbola.

$$R=\Delta\div\Sigma=1, \quad h=-4, \quad k=0.$$

Hence, the first transformed equation is

$$x^2+y^2-5xy=1. \tag{2}$$

In the second transformation,

$$\theta=45°, \quad P=-\tfrac{3}{2}, \quad Q=\tfrac{7}{2}.$$

Hence, the equation of the curve referred to its own axes is

$$3x^2-7y^2=-2,$$

from which we see that $a=\sqrt{\frac{2}{7}}$, $b=\sqrt{\frac{2}{3}}$, and a, or the transverse axis, lies on the axis of y.

To construct the equation, draw the axes O_1X_1, O_1Y_1, locate the centre $(-4, 0)$, and through it draw the third set

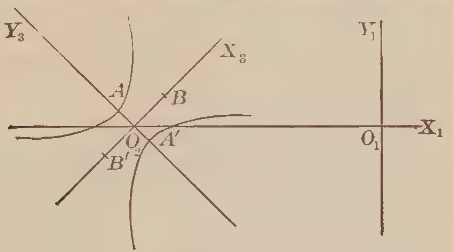


Fig. 76.

of axes O_2X_3, O_2Y_3, making $X_1O_2X_3 = 45°$. Then lay off $O_2A' = O_2A = \sqrt{\frac{2}{7}}$, and $O_2B = O_2B' = \sqrt{\frac{2}{3}}$; and draw the hyperbola having AA' and BB' as its transverse and conjugate axes respectively.

3. Determine the nature of the locus

$$x^2 + 2xy - y^2 + 8x + 4y - 8 = 0, \qquad (1)$$

transform the equation and construct it.

Here $\Sigma = -8$, $\Delta = -176$; hence, the locus is an hyperbola.

$$R = 22,\ h = -3,\ k = -1;$$

hence, the first transformed equation is

$$x^2 + 2xy - y^2 = 22. \qquad (2)$$

$$\theta = 22\tfrac{1}{2}°,\ P = \sqrt{2},\ Q = -\sqrt{2};$$

hence, the equation of the curve referred to its own axes is

$$\sqrt{2}x^2 - \sqrt{2}y^2 = 22. \qquad (3)$$

The hyperbola is equilateral, and its transverse axis lies on the axis of x. (Let the reader construct it.)

4. Determine the nature of the locus

$$x^2 + y^2 - 2xy + 2x - y - 1 = 0, \qquad (1)$$

transform the equation and construct it.

Here $\Sigma = 0$, Δ is not zero.

Therefore, the locus is a parabola.

$$\theta = 45^\circ,\ P = 0,\ Q = 2,\ U = \tfrac{1}{2}\sqrt{2},\ V = -\tfrac{3}{2}\sqrt{2};$$

hence, by revolving the axes through an angle of 45°, the equation becomes

$$2y^2 + \tfrac{1}{2}\sqrt{2}x - \tfrac{3}{2}\sqrt{2}y - 1 = 0,$$

or $$y^2 - \tfrac{3}{4}\sqrt{2}y + (\tfrac{3}{8}\sqrt{2})^2 = -\tfrac{1}{4}\sqrt{2}x + \tfrac{25}{32},$$

or $$(y - \tfrac{3}{8}\sqrt{2})^2 = -\tfrac{1}{4}\sqrt{2}(x - \tfrac{25}{16}\sqrt{2}). \qquad (2)$$

Passing to parallel axes whose origin is $(\tfrac{25}{16}\sqrt{2},\ \tfrac{3}{8}\sqrt{2})$, (2) becomes

$$y^2 = -\tfrac{1}{4}\sqrt{2}x, \qquad (3)$$

the locus of which is a parabola whose latus rectum, or parameter, is $\tfrac{1}{4}\sqrt{2}$.

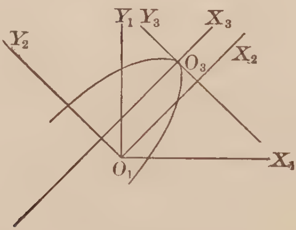


Fig. 77.

To construct the equation, draw the original axes O_1X_1, O_1Y_1; then draw the second set of axes O_1X_2, O_1Y_2, making

$$X_1O_1X_2 = 45^\circ.$$

Locate the new origin O_3, $(\tfrac{25}{16}\sqrt{2},\ \tfrac{3}{8}\sqrt{2})$, and through it draw the third set of axes O_3X_3, O_3Y_3, to which (3) refers the locus, which is now easily drawn.

5. Determine the nature of the locus

$$x^2 - 4xy + 4y^2 - 6x + 12y = 0.$$

Here $\Sigma = 0$ and $\Delta = 0$.

Factoring the first member of the equation, or solving the equation for x or y, we obtain as the equations of these lines

$$x - 2y = 0,\ x - 2y - 6 = 0.$$

Hence, the locus is two parallel straight lines.

6. Determine the nature of the locus of

$$y^2 - xy - 6x^2 - 3x + y = 0.$$

Here Σ is negative, and $\Delta = 0$.

Hence, the locus is two intersecting straight lines. Their equations are

$$y + 2x + 1 = 0, \text{ and } y - 3x = 0.$$

195. A **Conic** is the locus of a point whose distance from a fixed point bears a constant ratio to its distance from a fixed straight line.

196. *To find the equation of a conic.*

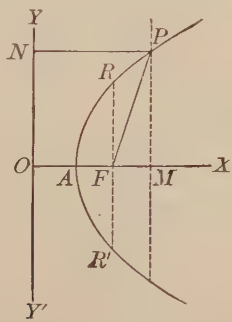


Fig. 78.

Let F be the fixed point, and YY' the fixed straight line. Through F draw XO perpendicular to YY', and use OX and OY as axes of reference. Let p denote the distance OF, and

e the constant ratio; then $FP \div NP = e$, P being any point (x, y) in the curve.

Now $\qquad \overline{FP}^2 = \overline{FM}^2 + \overline{MP}^2.$

But $\qquad FP = e \times NP = ex,$

$$FM = x - p,\ MP = y.$$

Therefore, $e^2x^2 = (x-p)^2 + y^2,$

or $$(1-e^2)\, x^2 + y^2 - 2px + p^2 = 0, \qquad (1)$$

which is the equation required.

Cor. In equation (1), which is of the second degree,

$$\Sigma = 4\,(1-e^2), \text{ and } \Delta = 4p^2e^2.$$

Hence, when the fixed point is without the fixed line, Δ is not zero, and

If $e < 1$, $\Sigma > 0$, and the conic is an ellipse.

If $e = 1$, $\Sigma = 0$, and the conic is a parabola.

If $e > 1$, $\Sigma < 0$, and the conic is an hyperbola.

When the fixed point is in the fixed line, $\Delta = 0$, and

If $e < 1$, $\Sigma > 0$, and the conic is the point $(0, 0)$.

If $e = 1$, $\Sigma = 0$, and the conic is a right line.

If $e > 1$, $\Sigma < 0$, and the conic is two intersecting right lines.

If $e = 0$, by § 61, the conic is a circle or a point, according as p is not or is zero.

From §§ 92, 150, 183, it follows that the fixed point is a **Focus,** the fixed right line a **Directrix,** and the constant ratio the **Eccentricity,** of the conic.

Exercise 40.

Determine the nature of the following loci, transform their equations, and construct them:

1. $3x^2 + 2y^2 - 2x + y - 1 = 0.$
2. $3x^2 + 2xy + 3y^2 - 16y + 23 = 0.$

3. $x^2 - 10xy + y^2 + x + y + 1 = 0.$

4. $x^2 + xy + y^2 + x + y - 5 = 0.$

5. $y^2 - x^2 - y = 0.$

6. $1 + 2x + 3y^2 = 0.$

7. $y^2 - 2xy + x^2 - 8x + 16 = 0.$

8. $x^2 - 2xy + y^2 - 6x - 6y + 9 = 0.$

9. $y^2 - 2x - 8y + 10 = 0.$

10. $4x^2 + 9y^2 + 8x + 36y + 4 = 0.$

11. $52x^2 + 72xy + 73y^2 = 0.$

12. $9y^2 - 4x^2 - 8x + 18y + 41 = 0.$

13. $y^2 - xy - 5x + 5y = 0.$

CHAPTER IX.

HIGHER PLANE CURVES.

197. An **Algebraic Curve** is one whose rectilinear equation contains only algebraic functions. A **Transcendental Curve** is one whose rectilinear equation involves other than algebraic functions. Thus, the loci of $y = a^x$, $y = \tan x$, $y = (a - x) \tan (\frac{1}{2}\pi x \div a)$, $y = \sin^{-1} x$ are transcendental curves. Transcendental curves and all algebraic curves above the second order are called **Higher Plane Curves.**

Let the symbol $F(x, y)$ denote any rational integral function of x and y, of the third or higher degree. If $F(x, y)$ breaks up into simple or quadratic factors in x and y, the locus of $F(x, y) = 0$ consists of lines of the first or second order. If $F(x, y)$ does not break up into rational factors in x and y, the locus of $F(x, y) = 0$ is a higher plane curve whose order is of the degree of the equation. Thus, the locus of

$$y^3 - x^3 = (y - x)(y^2 + xy + x^2) = 0,$$

consists of the right line $y - x = 0$ and the ellipse $y^2 + xy + x^2 = 0$; the locus of

$$y^4 + x^2y^2 + 2y^2 - 2x^4 - 5x^2 - 3 = (y^2 + 2x^2 + 3)(y^2 - x^2 - 1) = 0,$$

consists of the ellipse $y^2 + 2x^2 + 3 = 0$, and the hyperbola $y^2 - x^2 - 1 = 0$; while the locus of $y^3 - ax + x^2 = 0$ is a higher plane curve of the third order.

In this chapter we shall consider a few of the higher plane curves, some of which possess historic interest from the labor bestowed on them by the ancient mathematicians.

198. The Cissoid of Diocles. Let XH (Fig. 79) be a tangent to the circle XSO at the vertex of any diameter

OX; let OR be any right line from O to XH cutting the circle at S, and take $OP = RS$; then the locus of P, as OR revolves about O, is the *Cissoid.*

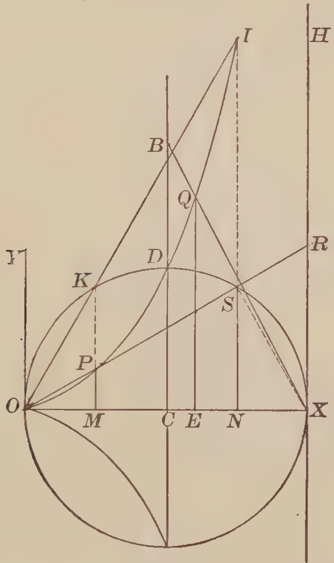


Fig. 79.

To find its equation referred to the rectangular axes OX and OY, let $OM = x$, $MP = y$, and $OC = CX = CD = r$.

Now, $\quad MP : OM :: NS : ON. \quad (1)$

Since $\quad OP = RS, \; OM = NX.$

Hence, $\quad ON = OX - NX = OX - OM = 2r - x,$

and $\quad NS = \sqrt{ON \times NX} = \sqrt{(2r - x)x}.$

Substituting these values in (1), we obtain

$$\frac{y}{x} = \frac{\sqrt{(2r - x)x}}{2r - x}, \text{ or } y^2 = \frac{x^3}{2r - x}, \qquad [48]$$

which is the equation sought.

A discussion of [48] leads to the following conclusions:

(i) The curve lies between the lines $x=0$ and $x=2r$.

(ii) It is symmetrical with respect to the axis of x.

(iii) It passes through the extremities of the diameter perpendicular to OX.

(iv) It has two infinite branches to which $x=2r$ is an asymptote.

Cor. 1. To find the polar equation, let $\theta=XOP$ and $\rho=OP$;

then $$\cos\theta=\frac{OX}{OR} \text{ or } \frac{OS}{OX}.$$

Hence, $$\rho=OP=SR=OR-OS=\frac{2r}{\cos\theta}-2r\cos\theta$$
$$=2r\frac{\sin^2\theta}{\cos\theta}.$$

Therefore, $\rho=2r\sin\theta\tan\theta$,
which is the equation sought.

The cissoid was invented by Diocles, a Greek mathematician of the second century B.C., for the solution of the problem of *finding two mean proportionals*, of which the *duplication of the cube* is a particular case.

Cor. 2. To duplicate the cube, in Fig. 79, take $CB=2r$, and draw BX cutting the cissoid in Q; then, since $CX=\frac{1}{2}CB$, $EX=\frac{1}{2}EQ$. But from the equation of the cissoid, we have
$$\overline{EQ}^2=\frac{\overline{OE}^3}{EX}=\frac{\overline{OE}^3}{\frac{1}{2}EQ}; \text{ therefore, } \overline{EQ}^3=2\overline{OE}^3.$$

Let c denote the edge of any given cube; take c_1 so that
$$OE : EQ :: c : c_1, \text{ or } \overline{OE}^3 : \overline{EQ}^3 :: c^3 : c_1^3.$$

But $$\overline{EQ}^3=2\overline{OE}^3; \text{ therefore, } c_1^3=2c^3;$$
that is, c_1 is the edge of a cube double the given cube in volume. In like manner, by taking $CB=mr$, we can find the edge of a cube m times the given cube in volume.

199. The Conchoid of Nicomedes. The **Conchoid** is the locus of a point P such that its distance from a fixed line XX', measured along the line through P and a fixed point A, is constant. A is the **Pole,** XX' the **Directrix,** and the constant distance RP, denoted by b, is the **Parameter.**

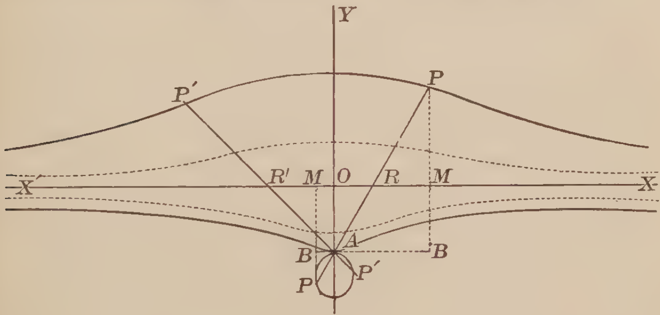


Fig. 80.

To construct the conchoid by points, through A draw any line AP cutting XX' in R. Lay off $RP=b$ on both sides of XX'. In like manner locate the points P', and any number of others, and trace the branches through them.

To find the equation of the curve referred to XX' and a line through A perpendicular to XX'; let

$$OM=x,\ MP=y,\ AO=a.$$

Now, $\quad AB : BP :: RM : MP,$

and $\quad RM=\sqrt{\overline{RP}^2-\overline{MP}^2}=\sqrt{b^2-y^2}.$

Therefore, $\quad x : y+a :: \sqrt{b^2-y^2} : y.$

Therefore, $\quad x^2y^2=(y+a)^2\,(b^2-y^2),\qquad [49]$

which is the equation of both branches of the conchoid. A discussion of [49] leads to the following conclusions:

(i) The curve lies between the lines $y=b$ and $y=-b$.

(ii) The curve is symmetrical with respect to the axis of y.

(iii) The axis of x is an asymptote to each branch of the curve.

If $b > a$, the lower branch has an oval or loop, as in the figure.

If $b = a$, the lower branch passes through A and is somewhat like that in the figure, without the loop below A.

If $b < a$, the upper and lower branches are like the dotted lines in the figure.

If $a = 0$, the conchoid becomes a circle.

Cor. 1. If A is the pole and AY the polar axis, we have

$$\rho = AR' \pm R'P' = a \sec \theta \pm b,$$

which is the polar equation of the conchoid.

The conchoid, invented by Nicomedes, a Greek mathematician of the second century B.C., was, like the cissoid, first formed for solving the problem of *finding two mean proportionals* or *duplicating the cube.* It is more readily applicable, however, to the *trisection of an angle*, a problem not less celebrated among the ancients.

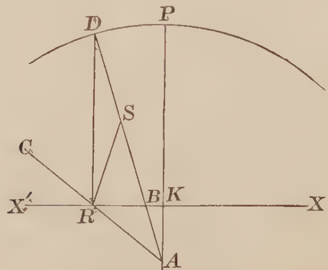


Fig. 81.

Cor. 2. To trisect any angle, as CAP (Fig. 81), on AC lay off AR any length; through R draw RK perpendicular to AP, and take $KP = 2AR$. Construct a conchoid with A as a pole, XX' as a directrix, and KP as a parameter.

At R erect a perpendicular to XX' intersecting the conchoid in D; then DA will trisect the angle RAP.

For bisect DB in S; then

$$RS = SD = \tfrac{1}{2}KP = AR.$$

Therefore, $\angle RAS = \angle RSA = 2\angle RDS = 2\angle DAP$.
Therefore, $\angle DAP = \tfrac{1}{3}\angle RAP$.

200. The Lemniscate of Bernoulli. The *Lemniscate* is the locus of the intersection of a tangent to a rectangular hyperbola with the perpendicular to it from the centre.

To find its equation we proceed as follows:

The equation of the tangent to the equilateral hyperbola $x^2 - y^2 = a^2$ at the point (x_1, y_1) is

$$x_1x - y_1y = a^2. \tag{1}$$

The equation of the perpendicular from the origin to (1) is

$$y = -\frac{y_1}{x_1}x, \text{ or } \frac{x}{x_1} = -\frac{y}{y_1}. \tag{2}$$

Solving (1) and (2) for x_1 and y_1 by multiplying each term of (1) by one of the members of (2), we obtain

$$x^2 + y^2 = \frac{a^2x}{x_1} = -\frac{a^2y}{y_1}.$$

Therefore, $x_1 = \dfrac{a^2x}{x^2+y^2}$, $y_1 = -\dfrac{a^2y}{x^2+y^2}$.

But, since (x_1, y_1) is on the hyperbola, we have $x_1{}^2 - y_1{}^2 = a^2$; hence, by substitution, we obtain

$$(x^2 + y^2)^2 = a^2(x^2 - y^2), \tag{50}$$

which is the equation sought.

From [50] it follows that the curve is symmetrical with respect to both axes. The form of the curve is given in Fig. 82.

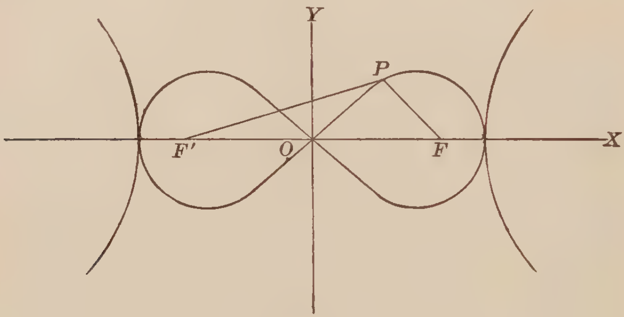


Fig. 82.

Cor. 1. Substituting $\rho \cos \theta$ for x, and $\rho \sin \theta$ for y in [50], and remembering that $\cos^2 \theta - \sin^2 \theta = \cos 2\theta$, we obtain

$$\rho^2 = a^2 \cos 2\theta \tag{2}$$

as the polar equation of the lemniscate.

Cor. 2. From (2), $\rho = \pm a \sqrt{\cos 2\theta}$.

Hence, when $\theta = 0$, $\rho = \pm a$. As θ increases from 0° to 45°, ρ changes from $\pm a$ to ± 0, and the portions in the first and third quadrants are traced. As θ increases from 45° to 135°, $\cos 2\theta$ is negative and ρ is imaginary. As θ increases from 135° to 180°, ρ changes from ± 0 to $\pm a$, and the portions in the second and fourth quadrants are traced. Therefore,

(i) The curve consists of two ovals meeting at the pole O.

(ii) The tangents to the curve at O are the asymptotes to the equilateral hyperbola.

201. The Witch of Agnesi. Let YH be a tangent to the circle OKY at the vertex of the diameter OY; let

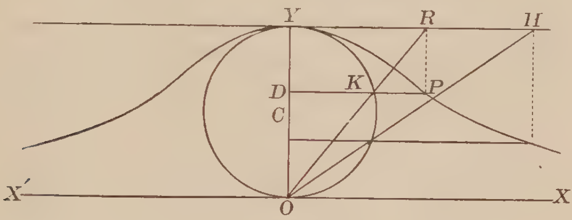


Fig. 83.

OR be any line from O to YH cutting the circle in K; produce the ordinate DK, and make $DP = YR$; then the locus of P is the *Witch.*

To find its equation, let the tangent OX and the diameter OY be the axes; let $OY = 2r$ and P be any point; then $OD = y$, $DP = x$, and

$$OD : OY :: DK : DP (= YR), \tag{1}$$

or

$$y : 2r :: \sqrt{y(2r - y)} : x.$$

Therefore, $\boldsymbol{x^2 y = 4r^2(2r - y)}$ [51]

is the equation of the Witch.

Cor. 1. Since $x = \pm 2r\sqrt{(2r - y) \div y}$, it follows that

(i) The curve is symmetrical with respect to the axis of y.

(ii) The curve lies between the lines $y = 0$ and $y = 2r$.

(iii) The axis of x is an asymptote to each infinite branch.

Cor. 2. From (1) it follows that corresponding abscissas of the circle and the Witch are proportional to the ordinate and the diameter of the circle.

The Witch was invented by Donna Maria Agnesi, an Italian mathematician of the eighteenth century.

202. The Cycloid. A Cycloid is generated by a point P in the circumference of a circle RPC, which rolls along a right line OX. The curve consists of an unlimited number of branches, but a single branch is usually termed a *cycloid.* The right line OX is called the **Base**; the rolling circle RPC the **Generatrix**; and P the **Generating Point.** If $OK = KX$, the perpendicular KH is the **Axis,** and H the **Highest Point.**

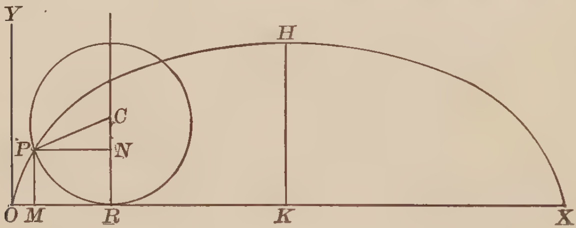


Fig. 84.

To find the equation of the curve, first take as the axis of X the base OX, and as the origin the point O, where the curve meets the base. Let r denote the radius of the generatrix RPC, and θ the angle PCR; then arc PR equals OR over which it has rolled, and $\theta = \text{arc } PR \div r$. Denote the coördinates of P by x and y; then

$$x = OM = OR - MR = \text{arc } PR - PN = r\,\theta - r \sin \theta.$$

$$y = MP = RC - NC = r - r \cos \theta.$$

Therefore,

$$\left.\begin{aligned} x &= r\,(\theta - \sin \theta) \\ y &= r\,(1 - \cos \theta) \end{aligned}\right\}. \qquad (1)$$

Equations (1) taken as simultaneous are the equations of the cycloid.

To eliminate θ between these equations, from the second, we obtain

$$\cos\theta = \frac{r-y}{r};\ \text{therefore, } \sin\theta = \pm\frac{\sqrt{2ry-y^2}}{r};$$

and $$\operatorname{vers}\theta = [1-\cos\theta] = \frac{y}{r};\ \text{or } \theta = \operatorname{vers}^{-1}\frac{y}{r}.$$

Substituting these values of θ and $\sin\theta$ in the first of equations (1), we have

$$\boldsymbol{x = r\operatorname{vers}^{-1}\frac{y}{r} \mp \sqrt{2ry-y^2},} \qquad [52]$$

the equation of the cycloid in the more common form.

In the value of $\sin\theta$, and therefore in equation [52], the upper or lower sign is used according as $\theta <$ or $> \pi$; that is, according as the point is on the first or second half of a branch.

From $y = r(1-\cos\theta)$ it follows that the locus lies between the lines $y=0$ and $y=2r$.

For $y=0$ in [52], $x = 0, \pm 2\pi r, \pm 4\pi r, \ldots\ldots$; hence, the locus consists of an unlimited number of branches like OHX, both to the right and to the left of OY.

203. Let the highest point O (Fig. 85) be taken as the origin, and OX parallel to the base as the axis of x; then OY, the axis of the curve, will be the axis of y. Let θ denote the angle HCK.

The point K was at Y when P was at O, and arc $KH = YH$. Hence,

$$x = OM = YH + BP = r\theta + r\sin\theta.$$
$$y = -MP = -NC + BC = -r + r\cos\theta.$$

Hence, the equations are

$$\left.\begin{aligned} x &= r(\theta + \sin\theta) \\ y &= r(\cos\theta - 1) \end{aligned}\right\}, \qquad (1)$$

or $$x = r\operatorname{vers}^{-1}\left(\frac{-y}{r}\right) \pm \sqrt{-2ry-y^2}. \qquad (2)$$

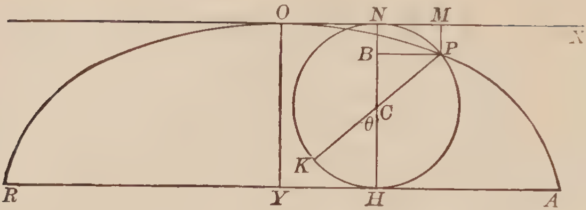


Fig. 85.

The invention of the cycloid is usually ascribed to Galileo. After the conics, no curve has exercised the ingenuity of mathematicians more than the cycloid, and their labors have been rewarded by the discovery of a multitude of interesting properties. Thus, the length of the branch ROA is eight times the radius, and the area ROA is three times the area of the generating circle.

Exercise 41.

1. Prove that the cissoid is the locus of the intersection of a tangent to the parabola $y^2=-8rx$ with the perpendicular to it from the origin.

By § 99, the equation of a tangent to $y^2=-8rx$ is

$$y=mx-\frac{2r}{m}, \tag{1}$$

and the perpendicular to it from (0, 0) is

$$y=-\frac{1}{m}x\text{; therefore, } m=-\frac{x}{y}. \tag{2}$$

Eliminating m between (1) and (2), we obtain

$$y^2=\frac{x^3}{2r-x}.$$

2. At the centre of any circle C (Fig. 86), erect $CH\perp$ to the diameter OX; and on XO produced lay off $OA=OC=r$. Let LQR be a rectangular ruler of which the leg QR equals $2r$. If the ruler is moved so that the leg LQ

passes through A, while the end R slides along CH, prove that the locus of P, the middle point of QR, is a cissoid.

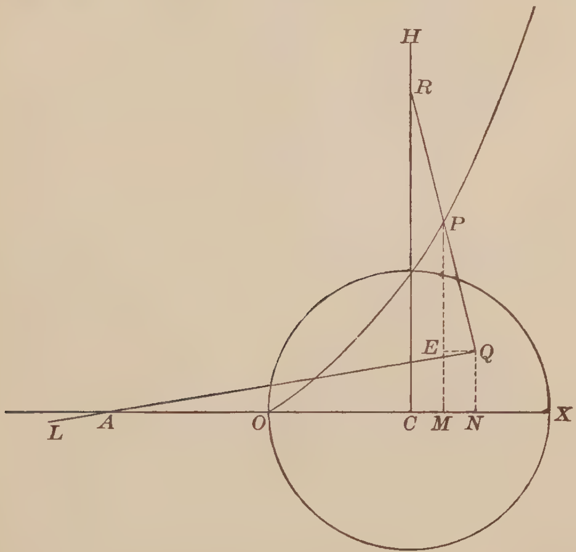


Fig. 86.

Let OX be the axis of x, O the origin, and AQR any position of the ruler; then $OM = x$, $MP = y$,

$$EQ = MN = CM = x - r, \quad AN = 2OM = 2x.$$

$$\overline{EP}^2 = \overline{PQ}^2 - \overline{EQ}^2 = r^2 - (x-r)^2 = 2rx - x^2. \quad (1)$$

$$y = MP = NQ + EP. \quad (2)$$

$$NQ : AN :: EQ : EP,$$

or
$$NQ : 2x :: x - r : \sqrt{2rx - x^2}. \quad (3)$$

From (1), (2), and (3), we obtain

$$y = \frac{2x(x-r)}{\sqrt{2rx - x^2}} + \sqrt{2rx - x^2}, \text{ or } y^2 = \frac{x^3}{2r - x}.$$

This method of describing the cissoid by continuous motion was invented by Sir Isaac Newton.

3. In Fig. 79 prove that NS and ON are two mean proportionals between OM and NI; that is, prove

$$OM : NS :: ON : NI.$$

$$\overline{NS}^2 = NX \times ON = OM \times ON.$$

The right line OI will pass through K; hence,

$$\angle OIN = \angle YOI = \tfrac{1}{2} \text{ arc } OK = \tfrac{1}{2} \text{ arc } SX$$
$$= \angle SOX;$$

therefore, $NS : ON :: ON : NI$, etc.

4. If in the lemniscate (Fig. 82) $OF' = OF = \frac{1}{2} a \sqrt{2}$, prove that $FP \times F'P$ is constant, P being any point on the curve; and hence that the lemniscate may be defined as the locus of a point the product of whose distances from two fixed points is constant.

5. Construct the logarithmic curve $y = a^x$, or $x = \log_a y$. Prove that every logarithmic curve passes through the point (0, 1), and has the axis of x as an asymptote.

6. The **Trochoid** is the curve traced by any point in the radius of a circle rolling on a right line. If r denotes the radius of the circle, b the distance of the generating point from its centre, and θ denotes the same angle as in § 202, show that the equations of the trochoid are

$$\left.\begin{aligned} x &= r\theta - b \sin \theta \\ y &= r - b \cos \theta \end{aligned}\right\}. \tag{1}$$

When $b < r$, the trochoid is called the **Prolate Cycloid**; and when $b > r$, the **Curtate Cycloid.** When $b = r$, the curve is the *cycloid*.

Spirals.

204. A **Spiral** is the locus of a point whose radius vector continually increases, or continually decreases, while its vectorial angle increases (or decreases) without limit.

A **Spire** is the portion of the spiral traced during one revolution of the radius vector.

The **Measuring Circle** is the circle whose centre is the pole and whose radius is the value of ρ when $\theta = 2\pi$.

205. The Spiral of Archimedes. If the radius vector of a moving point has a constant ratio to its vectorial angle, that is, if

$$\rho = c\theta, \tag{1}$$

the locus is the Spiral of Archimedes.

From equation (1), we have

when $\theta = 0,\ \frac{1}{4}\pi,\ \frac{2}{4}\pi,\ \frac{3}{4}\pi,\ \pi,\ \frac{5}{4}\pi,\ \frac{6}{4}\pi,\ \frac{7}{4}\pi,\ 2\pi,\ \frac{9}{4}\pi,\ \ldots\ldots$

$\rho = 0,\ \frac{1}{4}\pi c,\ \frac{2}{4}\pi c,\ \frac{3}{4}\pi c,\ \pi c,\ \frac{5}{4}\pi c,\ \frac{6}{4}\pi c,\ \frac{7}{4}\pi c,\ 2\pi c,\ \frac{9}{4}\pi c,\ \ldots\ldots$

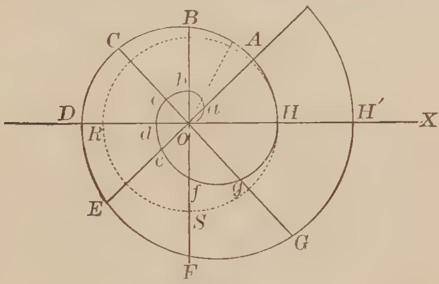


Fig. 87.

Hence, to construct the Spiral of Archimedes, draw the radial lines OH', OA, OB, ... OG, including angles of $\frac{1}{4}\pi$; on these lay off $Oa = \frac{1}{4}\pi c$, $Ob = \frac{2}{4}\pi c$, $Oc = \frac{3}{4}\pi c$, ..., $OH = 2\pi c$, and trace the first spire $OabcdefgH$ through these points. Any number of other spires are easily constructed by noting that the distance between two spires, measured on a radius vector, is equal to $2\pi c$. Thus, by taking aA, bB, cC, dD, eE, fF, gG, HH', each equal to $2\pi c$, we obtain points of the second spire; and so on. Any number of additional radial lines may be drawn to locate points in the curve.

The spires, being everywhere equally distant along radial lines, are said to be parallel. The measuring circle is HRS, whose radius is OH or $2\pi c$.

206. The Reciprocal or Hyperbolic Spiral. If the radius vector of a point varies inversely as its vectorial angle, that is, if

$$\rho\theta = c, \tag{1}$$

the locus is the **Reciprocal Spiral.**

Since $\rho = c \div \theta$, we have

$$\text{when } \theta = 0, \quad \tfrac{1}{4}\pi, \quad \tfrac{1}{2}\pi, \quad \tfrac{3}{4}\pi, \quad \pi, \quad \tfrac{5}{4}\pi, \quad \tfrac{3}{2}\pi, \quad \tfrac{7}{4}\pi, \quad 2\pi, \ldots\ldots$$

$$\rho = \infty, 8\frac{c}{2\pi}, 4\frac{c}{2\pi}, \tfrac{8}{3}\frac{c}{2\pi}, 2\frac{c}{2\pi}, \tfrac{8}{5}\frac{c}{2\pi}, \tfrac{4}{3}\frac{c}{2\pi}, \tfrac{8}{7}\frac{c}{2\pi}, \frac{c}{2\pi}, \ldots\ldots$$

Hence, the radius of the measuring circle is $c \div 2\pi$, and its circumference is c.

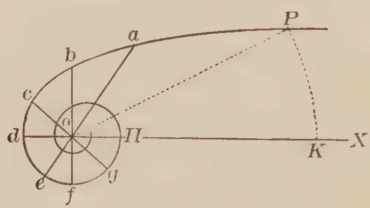


Fig. 88.

To construct the curve, draw the radial lines OX, Oa, Ob, Oc, Od, Oe, Of, Og, including angles of $\frac{1}{4}\pi$. Take $OH = c \div 2\pi$, and lay off $Oa = 8 \times OH$, $Ob = 4 \times OH$, $Oc = \frac{8}{3} \times OH$, $Od = 2 \times OH$, etc.; and through the points a, b, c, d, etc., trace the curve. In like manner, any number of spires may be drawn. From (1) it is evident that ρ approaches zero, as θ approaches infinity; that is, the curve continually approaches the pole without ever reaching it.

Since $\rho\theta = c$, it follows that the arc PK described with the radius vector of any point P is constant and equal to c.

Now, as ρ approaches infinity, this arc approaches a perpendicular to OX. Hence, the line parallel to OX at the distance of c above it is an asymptote to the infinite branch of the spiral.

207. The Lituus. If the square of the radius vector of a point varies inversely as its vectorial angle, that is, if

$$\rho^2\theta = c,$$

the locus is the **Lituus.**

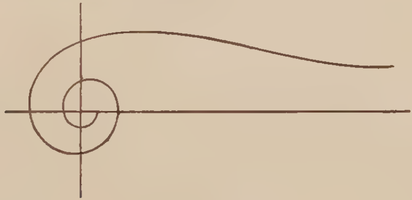


Fig. 89.

Let the student construct the curve from its equation and show that

(i) The curve continually approaches the pole without ever reaching it, as θ increases without limit.

(ii) The polar axis is an asymptote to the infinite branch.

208. The Logarithmic Spiral. If the radius vector of a point increases in a geometrical ratio, while its vectorial angle increases in an arithmetical ratio; that is, if

$$\rho = a^\theta, \text{ or } \theta = \log_a \rho, \tag{1}$$

the locus is the **Logarithmic Spiral.**

Since $\rho = 1$ when $\theta = 0$, every logarithmic spiral passes through the point (1, 0).

To construct a logarithmic spiral, let $a = 2$; then $\rho = 2^\theta$.

When $\theta = 0$, $1 (= 57.3°)$, $2 (= 114.6°)$, ..., 2π.

$\rho = 1$, 2 , 4 , ..., 77.88.

In Fig. 90 let $XOb = 57.3°$, $XOc = 114.6°$, $Oa = 1$. $Ob = 2$, $Oc = 4$, then a, b, c are three points on the spiral. As θ increases, ρ increases rapidly, but it becomes infinity only when θ does; and hence only after an infinite number of

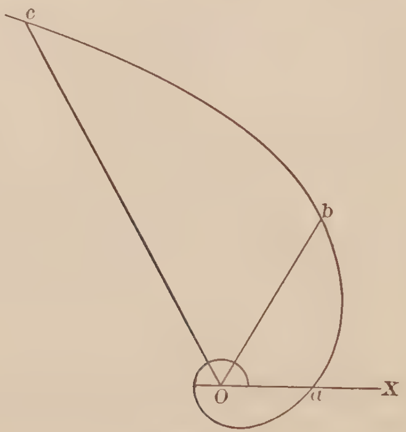


Fig. 90.

revolutions. As θ decreases from zero, ρ decreases from unity. Since ρ approaches zero as θ approaches negative infinity, the curve approaches the pole without ever reaching it.

209. The Parabolic Spiral. If in the equation $y^2 = 4px$, the values of x are laid off from A (Fig. 91) on the circle AH, and those of y on its corresponding radii produced, the locus of the point thus determined is the **Parabolic Spiral.**

To find its equation, denote the radius OA by r, and let P be any point; then

$$x = AmH = r\theta.$$

$$y = HP = OP - OH = \rho - r.$$

Substituting these values of x and y in $y^2 = 4px$, we obtain the polar equation

$$(\rho - r)^2 = 4pr\,\theta.$$

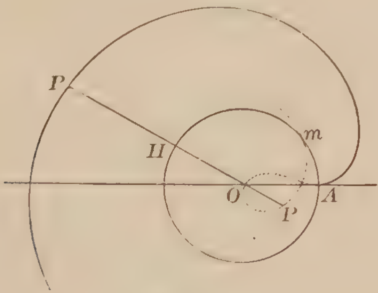


Fig. 91.

The curve consists of two branches beginning at A; the one determined by the positive values of y is an infinite spiral lying entirely without the circle; the other branch passes through the pole, forms a loop, and passes without the circle when $\rho = -r$, and $\theta = r \div p$.

NOTE. Among the ancients no problems were more celebrated than the "Duplication of the Cube" and the "Trisection of an Angle." Hippocrates of Chios reduced these two problems to the more general problem of *finding two mean proportionals between two given lines.* Thus, if c is the edge of the given cube, and x and y are the two mean proportionals between c and $2c$, we have

$$c : x = x : y = y : 2c.$$

Therefore, $$\left(\frac{c}{x}\right)^3 = \frac{c}{x} \times \frac{x}{y} \times \frac{y}{2c} = \tfrac{1}{2}, \text{ or } x^3 = 2c^3.$$

Hence, x, the first of the two mean proportionals between c and $2c$, is the edge of a cube double the given cube in volume.

After years of vain endeavor to solve these problems by the right line and the circle, the ancient geometers began to invent and study other curves, as the conics and some of the higher plane curves. The invention of the conics is credited to Plato, in whose school their properties were an object of special study.

PART II. — SOLID GEOMETRY.

CHAPTER I.

THE POINT.

210. The position of a point in space may be determined by referring it to three fixed planes meeting in a point. The fixed planes are called **Coördinate Planes,** their lines of intersection the **Coördinate Axes,** and their common point the **Origin**. In what follows we shall employ coördinate planes that intersect each other at right angles.

Let XOY, YOZ, ZOX, be three planes of indefinite extent intersecting each other at right angles in the lines

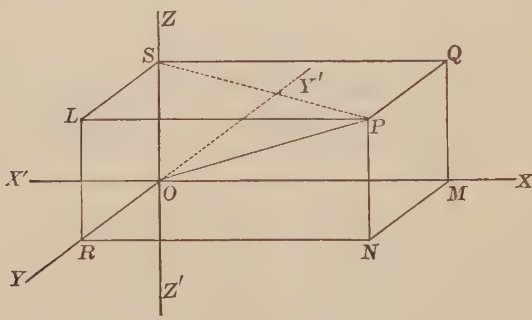


Fig. 92.

XX', YY', ZZ'. These coördinate planes are called the planes xy, yz, zx, respectively; the axes XX', YY', ZZ' are called the axes of x, y, z, respectively; and their common point O is the origin.

The coördinate planes divide all space into eight portions, called **Octants,** which are numbered as follows: The *First Octant* is O-XYZ, the *Second*, O-$YX'Z$, the *Third*, O-$X'Y'Z$, the *Fourth*, O-$Y'XZ$, the *Fifth*, O-XYZ', the *Sixth*, O-$YX'Z'$, the *Seventh*, O-$X'Y'Z'$, the *Eighth*, O-$Y'XZ'$. The fifth octant is below the first.

Let P be any point in space, and through it pass three planes parallel respectively to the three coördinate planes, thus forming the rectangular parallelopiped P-$ORNM$. The position of P will be determined when we know the lengths and directions of the lines LP, QP, NP. These three lines are called the **Rectangular Coördinates** x, y, z, of the point P, which is written (x, y, z).

A coördinate is *positive* when it has the direction of OX, OY, or OZ; hence, it is *negative* when it has the direction of OX', OY', or OZ'.

Thus the coördinate x is positive or negative according as it extends to the right or to the left from the plane yz; y is positive or negative according as it extends to the front or to the rear of the plane zx; and z, according as it extends upward or downward from the plane xy. Hence, the octant in which a point is situated is determined by the *signs* of its coördinates. Since the first octant has the positive directions of the axes for its edges, the coördinates of a point in the first octant are all positive. If (a, b, c) is a point in the first octant, the corresponding point

in the second octant is $(-a, b, c)$,
in the third octant is $(-a, -b, c)$,
in the fourth octant is $(a, -b, c)$,
in the fifth octant is $(a, b, -c)$,
in the sixth octant is $(-a, b, -c)$,
in the seventh octant is $(-a, -b, -c)$,
in the eighth octant is $(a, -b, -c)$.

The point $(x, y, 0)$ is in the plane xy.
The point $(x, 0, 0)$ is in the axis of x.
The point $(0, 0, 0)$ is the origin.

The lines OM, OR, OS, or OM, MN, NP, may be taken as the coördinates of P, for they have the same length and direction as LP, QP, NP, respectively. To construct P, (x, y, z), we take $OM = x$, draw MN parallel to OY, take $MN = y$, draw NP parallel to OZ, and take $NP = z$.

211. The **Radius Vector** of a point is the line drawn to it from the origin. Thus, OP $(= \rho)$ is the radius vector of P. From the rectangular parallelopiped in Fig. 92, we have

$$\overline{OP}^2 = \overline{OM}^2 + \overline{MN}^2 + \overline{NP}^2.$$

Hence, denoting the coördinates of P by x, y, z, we have

$$\rho^2 = x^2 + y^2 + z^2. \qquad [53]$$

That is, *the square of the radius vector of a point is equal to the sum of the squares of its rectangular coördinates.*

212. By the angle between two non-intersecting straight lines is meant the angle between any two intersecting straight lines that are parallel to them. Thus, any line parallel to OP (Fig. 92) makes the angles XOP, YOP, ZOP with the axes of x, y, z, respectively.

The angles which a line makes with the positive directions of the coördinate axes are called its **Direction Angles**; and the cosines of these angles are called the **Direction Cosines** of the line.

The direction angles of a line are always positive and cannot exceed π, or 180°.

213. Let α, β, γ denote the direction angles of OP (Fig. 92), or any line parallel to it, and x, y, z the coördinates of P, and let $\rho = OP$; then evidently

$$x = \rho \cos \alpha,\ y = \rho \cos \beta,\ z = \rho \cos \gamma. \qquad [54]$$

Squaring and adding [54], and substituting in [53], we obtain

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \qquad [55]$$

That is, *the sum of the squares of the direction cosines of a line is equal to unity.*

COR. Whatever are the values of x, y, z in [54], if each is divided by ρ, or $\sqrt{x^2+y^2+z^2}$, the quotients are the direction cosines of the radius vector of the point (x, y, z).

Hence, *if any three real quantities are each divided by the square root of the sum of their squares, the quotients will be the direction cosines of some line.*

Exercise 42.

1. In what octants may (x, y, z) be, when x is positive? when x is negative? when y is positive? when y is negative? when z is positive? when z is negative?

2. In what octant is $(-2, 4, 6)$? $(2, 4, -3)$? $(-2, 4, -1)$? $(-2, -3, -1)$? $(-2, -3, 3)$? $(2, -3, 1)$? $(2, -1, -3)$? Construct each point.

3. In what line is $(a, 0, 0)$? $(0, 0, c)$? $(0, b, 0)$?

4. In what plane is $(a, b, 0)$? $(a, 0, c)$? $(0, b, c)$?

5. Find the length of the radius vector of $(3, 4, 5)$, $(2, -3, -1)$, $(7, -3, -5)$. Find the direction cosines of the radius vector of each point.

6. The direction cosines of a line are proportional to 1, 2, 3; find their values. What is the direction of the line?

7. What is the direction of the line whose direction cosines are proportional to A, B, C? What are the values of its direction cosines?

8. Two direction angles of a line are 60° and 45°, what is the third? If two are 60° and 30°, what is the third? If two are 135° and 60°, what is the third?

214. Projections. The *projection* of a point upon a right line is the foot of the perpendicular from the point to the line; or it is the intersection of the line with the plane through the point perpendicular to the line. Thus M, R, S (Fig. 92) are the projections of the point P upon the axes of x, y, z, respectively. Here and in the following pages, by projection is meant the orthogonal projection.

The *projection* of a limited right line on another right line is the part intercepted between the projections of its extremities. Thus, OM, OR, OS (Fig. 92) are the projections of OP on the axes of x, y, z, respectively.

That is, *the coördinates of any point are the projections of its radius vector on the three axes.*

215. The projections of any line PQ on parallel lines are equal; for these projections are parallel lines included between parallel planes through P and Q. Now the projection of any straight line on another that passes through one of its extremities is evidently equal to the product of its length into the cosine of their included angle.

Hence, *the projection of any limited straight line on any other straight line is equal to its length multiplied by the cosine of the angle between the lines.*

216. Let AD (Figs. 93, 94) be a straight line, and $ABCD$ any broken line *in space*, connecting the points A and D, and let A', B', C', D' be the projections of A, B, C, D upon OX, whose positive direction is OX. Denote the angle RAD by ϕ, the lengths of the lines AB, BC, CD by l_1, l_2, l_3, and the angles which they make with the positive direction of OX by α_1, α_2, α_3; then in Fig. 93 we have

$$A'D' = A'B' + B'C' + C'D'.$$

Therefore, $\quad AD \cos \phi = l_1 \cos \alpha_1 + l_2 \cos \alpha_2 + l_3 \cos \alpha_3.$ [56]

In Fig. 94, $C'D'$ is negative; but $l_3 \cos \alpha_3$ is also negative, since α_3, or SCD, is obtuse; hence formula [56] holds true in all cases.

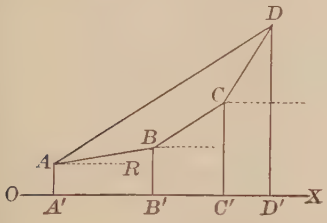


Fig. 93.

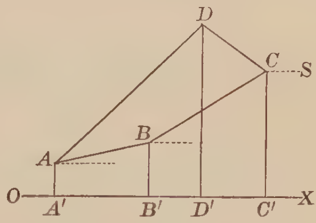


Fig. 94.

That is, *the algebraic sum of the projections on a given line of the parts of any broken line connecting any two points is equal to the projection on the same line of the straight line joining the same two points.*

217. *To find the angle between two straight lines in terms of their direction cosines.*

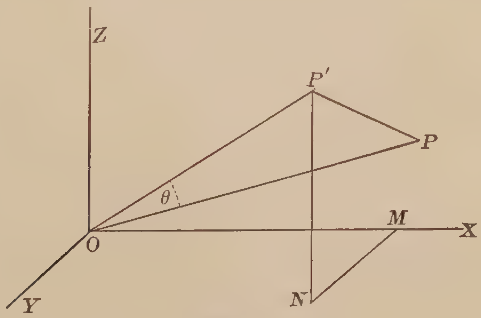


Fig. 95.

Let OP and OP' be parallel respectively to any two given lines in space. Let θ denote their included angle, and α, β, γ

and α', β', γ' their direction angles, respectively. Let OM, MN, NP' be the coördinates of P'; then the projection of OP' on OP equals the sum of the projections of OM, MN, NP' on OP; that is,

$$OP' \cos\theta = OM \cos\alpha + MN \cos\beta + NP' \cos\gamma.$$

But $\quad OM = OP' \cos\alpha', MN = OP' \cos\beta', NP' = OP' \cos\gamma'.$

Hence, $\quad OP' \cos\theta = OP' \cos\alpha' \cos\alpha + OP' \cos\beta' \cos\beta + OP' \cos\gamma' \cos\gamma,$

or $$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma', \qquad [57]$$

which is the required formula.

218. *To find the distance between two points in terms of their coördinates.*

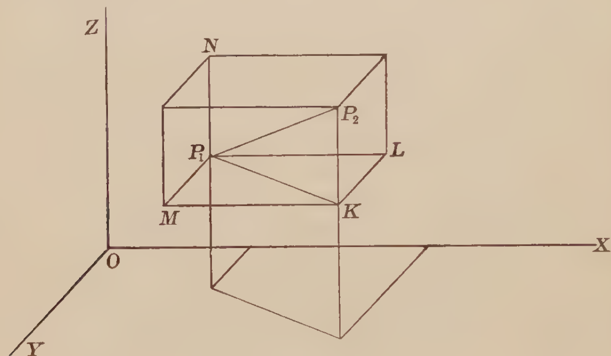


Fig. 96.

Let P_1 be the point (x_1, y_1, z_1), and P_2 the point (x_2, y_2, z_2). Through P_1 and P_2 pass planes parallel to the coördinate planes, thus forming the rectangular parallelopiped whose diagonal is P_1P_2, and whose edges P_1L, LK, KP_2 are parallel to the axes of x, y, z, respectively.

Then $$\overline{P_1P_2}^2 = \overline{P_1L}^2 + \overline{LK}^2 + \overline{KP_2}^2. \qquad (1)$$

Now P_1L is the difference of the distances of P_1 and P_2 from the plane yz, so that $P_1L = x_2 - x_1$. For like reason, $LK = y_2 - y_1$, and $KP_2 = z_2 - z_1$. Hence, denoting the distance P_1P_2 by D, we have, by substituting in (1),

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}, \qquad [58]$$

which is the required formula.

Cor. 1. Since P_1L, LK, KP_2 are equal to the projections of the line P_1P_2 on the coördinate axes, it follows, from [58], that

The square of any line is equal to the sum of the squares of its projections on the axes.

Cor. 2. If α, β, γ are the direction angles of the line P_1P_2, we have

$$x_2 - x_1 = D \cos \alpha,\ y_2 - y_1 = D \cos \beta,\ z_2 - z_1 = D \cos \gamma.$$

219. Polar Coördinates. Let XOY be a fixed plane, OX a fixed line in it, and OZ a perpendicular to it at the fixed point O. To P, any point in space, draw OP, and

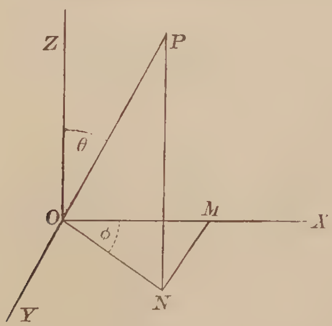


Fig. 97.

through OP pass a plane perpendicular to XOY, intersecting the latter in ON; then the distance OP and the angles ZOP and MON determine the point P, and are called its

Polar Coördinates; OP, denoted by ρ, is the **Radius Vector**; and the angles ZOP and MON, denoted by θ and ϕ, respectively, are the **Vectorial Angles.** The point P is written (ρ, θ, ϕ). ϕ determines the plane ZON, θ determines the line OP in that plane, and ρ locates P in OP.

COR. If XOY is a right angle, the rectangular coördinates of P are OM, MN, NP. To express these in terms of the polar coördinates of P, we have

$$x = OM = ON \cos\phi = OP \sin\theta \cos\phi = \rho \sin\theta \cos\phi.$$

$$y = MN = ON \sin\phi = OP \sin\theta \sin\phi = \rho \sin\theta \sin\phi.$$

$$z = NP = OP \cos\theta = \rho \cos\theta.$$

We readily obtain also

$$\rho = \sqrt{x^2 + y^2 + z^2},$$

$$\tan\theta = \frac{\sqrt{x^2 + y^2}}{z}, \quad \tan\phi = \frac{y}{x}.$$

220. The **Projection** of a point on a plane is the foot of the perpendicular from the point to the plane. The perpendicular itself is the **Projector** of the point. Thus, the point N (Fig. 97) is the projection of P on the plane xy, and PN is its projector.

The *projection* of a limited straight line on a plane is the straight line joining the projections of its extremities. The **Inclination** of a line to a plane is the angle it makes with its projection on that plane. The projection of a limited line is evidently equal to its length multiplied by the cosine of its inclination. Thus, $ON = OP \cos NOP$.

The *projection* of any curve upon a plane is the locus of the projections of all its points. The **Projecting Cylinder** of a curve is the locus of the projectors of all its points. In the case of a right line this locus is the **Projecting Plane.**

Exercise 43.

1. Find the distance between the points $(1, 2, 3), (2, 3, 4)$; $(2, 3, 4), (3, 4, 5)$; $(1, 2, 3), (3, 4, 5)$.

2. Prove that the triangle formed by joining the three points $(1, 2, 3), (2, 3, 1), (3, 1, 2)$ is equilateral.

3. The lengths of the projections of a line on the three coördinate axes are 3, 4, 5, respectively; find the length of the line.

4. Find the direction cosines of the radius vector of the point $(-3, -4, 5)$.

5. What lines have direction cosines proportional to $3, -2, -5$? Find the values of these direction cosines.

6. Find the angle between two straight lines whose direction cosines are proportional to 1, 2, 3 and 2, 3, 6, respectively.

7. Find the angle between two straight lines whose direction cosines are proportional to 1, 2, 3 and $5, -4, 1$, respectively.

8. Find the polar coördinates of the point $(\sqrt{3}, 1, 2\sqrt{3})$.

9. Find the rectangular coördinates of $(4, \frac{1}{6}\pi, \frac{1}{3}\pi)$.

10. If (x, y, z) bisects the line joining (x_1, y_1, z_1) to (x_2, y_2, z_2), prove that $x = \frac{1}{2}(x_1 + x_2)$, $y = \frac{1}{2}(y_1 + y_2)$, $z = \frac{1}{2}(z_1 + z_2)$.

11. If (x, y, z) divides the line joining (x_1, y_1, z_1) to (x_2, y_2, z_2), in the ratio $m_1 : m_2$, prove that

$$x = \frac{m_2 x_1 + m_1 x_2}{m_2 + m_1}, \quad y = \frac{m_2 y_1 + m_1 y_2}{m_2 + m_1}, \quad z = \frac{m_2 z_1 + m_1 z_2}{m_2 + m_1}.$$

12. Find the coördinates of the point that divides the line joining $(3, -2, 4)$ and $(1, 3, -2)$ in the ratio $1 : 3$.

13. Find the point that divides the line joining $(-2, -3, -1)$ and $(-5, -2, 4)$ in the ratio $5 : 2$.

CHAPTER II.

THE PLANE.

221. *To find the equation of a plane in terms of the length of the perpendicular from the origin, and its direction cosines.*

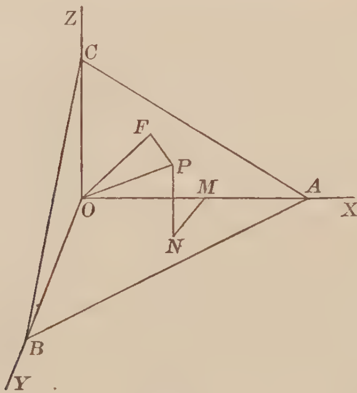


Fig. 98.

Let OF be the perpendicular to the plane ABC from the origin O; denote its length by p, and its direction angles by α, β, γ. Let P be any point in the plane, OP its radius vector, and OM, MN, NP its coördinates, x, y, z. Then the projection of OP on OF is equal to the sum of the projections of OM, MN, NP on OF. But as the plane is perpendicular to OF, p is the projection of OP on OF; and the projections of OM, MN, NP on OF are respectively $x \cos \alpha$, $y \cos \beta$, $z \cos \gamma$; hence,

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p, \qquad [59]$$

which is the equation required. Equation [59] is called the *normal* equation of a plane.

COR. 1. When the plane is perpendicular to one of the coördinate planes, the plane xy for example, OF lies in the plane xy; hence, $\gamma = \frac{1}{2}\pi$, $\cos\gamma = 0$, and equation [59] becomes

$$x \cos\alpha + y \cos\beta = p. \qquad (1)$$

COR. 2. When the plane is parallel to one of the coördinate planes, as the plane yz, OF lies in the axis of x; hence, $\cos\alpha = 1$, $\cos\beta = 0$, $\cos\gamma = 0$, and [59] becomes

$$x = p. \qquad (2)$$

COR. 3. Since OF is perpendicular to the plane ABC, and OX to YOZ, the dihedral angle $A\text{-}BC\text{-}O =$ angle FOX. For like reason, $B\text{-}CA\text{-}O = FOY$, and $C\text{-}BA\text{-}O = FOZ$.

222. *The locus of every equation of the first degree between three variables is a plane.*

A general form embracing every equation of the first degree between x, y, z is

$$Ax + By + Cz = D, \qquad (1)$$

in which D is positive.

Dividing both members of (1) by $\sqrt{A^2 + B^2 + C^2}$, we obtain

$$\frac{A}{\sqrt{A^2+B^2+C^2}}x + \frac{B}{\sqrt{A^2+B^2+C^2}}y + \frac{C}{\sqrt{A^2+B^2+C^2}}z$$
$$= \frac{D}{\sqrt{A^2+B^2+C^2}}, \qquad (2)$$

in which the coefficients of x, y, z are the direction cosines of some line (§ 213, Cor.). Thus (2) is in the form of [59] § 221; hence, the locus of (2), or (1), is a plane.

COR. 1. The length of the perpendicular from the origin upon plane (1) equals the second member of equation (2), and

the direction cosines of this perpendicular are the coefficients in (2) of x, y, z, respectively. These direction cosines are evidently proportional to A, B, C.

Hence, to construct equation (1), draw the radius vector of the point (A, B, C); the plane perpendicular to this line at the distance $\frac{D}{\sqrt{A^2+B^2+C^2}}$ from the origin is the locus of (1).

Cor. 2. To reduce any simple equation to the *normal form*, put it in the form of $Ax+By+Cz=D$, in which D is positive, and divide both members by $\sqrt{A^2+B^2+C^2}$.

Cor. 3. If a simple equation contains only two variables, its locus is perpendicular to the corresponding coördinate plane; if only one variable, its locus is perpendicular to the corresponding coördinate axis.

223. *To find the equation of a plane in terms of its intercepts on the axes.*

Let a, b, c denote respectively the intercepts on the axes of the plane whose equation is

$$Ax+By+Cz=D. \tag{1}$$

Making $y=z=0$, and therefore $x=a$, (1) becomes

$$Aa=D, \text{ or } A=D\div a.$$

Making $x=z=0$, and therefore $y=b$, (1) becomes

$$Bb=D, \text{ or } B=D\div b.$$

Making $x=y=0$, and therefore $z=c$, (1) becomes

$$Cc=D, \text{ or } C=D\div c.$$

Substituting these values in (1) and dividing by D, we have

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1, \tag{60}$$

which is the required equation. Equation [60] is called the *symmetrical* equation of a plane.

224. *To find the angle between any two planes.*

The angle included between the two planes

$$A'x + B'y + C'z = D',$$
$$Ax + By + Cz = D,$$

is evidently equal to the angle included between the perpendiculars to them from the origin. But the direction cosines of these perpendiculars are respectively (§ 222, Cor. 1),

$$\frac{A}{\sqrt{A^2+B^2+C'^2}},\ \frac{B}{\sqrt{A^2+B^2+C^2}},\ \frac{C}{\sqrt{A^2+B^2+C^2}},$$
$$\frac{A'}{\sqrt{A'^2+B'^2+C'^2}},\ \frac{B'}{\sqrt{A'^2+B'^2+C'^2}},\ \frac{C'}{\sqrt{A'^2+B'^2+C'^2}}.$$

Substituting these values in [57], we have

$$\cos\theta = \frac{AA' + BB' + CC'}{\sqrt{A^2+B^2+C^2}\sqrt{A'^2+B'^2+C'^2}}, \qquad [61]$$

in which θ equals the angle included between the planes.

Cor. 1. If the planes are parallel to each other, $\theta = 0$ and $\cos\theta = 1$. Putting $\cos\theta = 1$ in [61], clearing of fractions, squaring, transposing, and uniting, we obtain

$$(AB' - BA')^2 + (AC' - CA')^2 + (BC' - CB')^2 = 0.$$

Each term being a square, and therefore positive, this equation can be satisfied only when each term equals zero, giving us

$$AB' = BA',\ AC' = CA',\ BC' = CB',$$

or $$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'}.$$

Hence, *if two planes are parallel, the coefficients of x, y, z, in their equations, are proportional, and conversely.*

Cor. 2. *If the planes are perpendicular to each other,* $\cos\theta = 0$; *and hence,*

$$AA' + BB' + CC' = 0,$$

and conversely.

225. *To find the perpendicular distance of a given point from a given plane.*

Let the equation of the given plane be

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p, \tag{1}$$

and let (x_1, y_1, z_1) be the given point. Let the plane

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p', \tag{2}$$

which is evidently parallel to the given plane, pass through the given point (x_1, y_1, z_1); then we have

$$x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma = p'. \tag{3}$$

Hence, $\quad x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p = p' - p.$

But $p' - p$ equals numerically the distance between the planes (1) and (2), and is therefore the required distance.

Hence, to find the distance of any point from the plane

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p,$$

substitute the coördinates of the point for x, y, z in the expression

$$x \cos \alpha + y \cos \beta + z \cos \gamma - p.$$

Cor. If the equation of the plane is $Ax + By + Cz = D$, and d denotes the distance of (x_1, y_1, z_1) from this plane, we have

$$d = \frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}}.$$

The distance as given by the formulas will evidently be *positive* or *negative*, according as the point and origin are on opposite sides of the plane, or on the same side. The sign may be neglected if simply the numerical distance is required.

Exercise 44.

1. To which coördinate plane is $3y - 4z = 2$ perpendicular? $x - 8z - 7 = 0$? $x - 2y = 2$? $x = mz + p$? $y = nx + q$? What is the locus of $z = 5$? $y = -7$? $y = 4$? $z = -2$? $x = 0$? $y = 0$? $z = 0$?

2. Reduce to the normal form

$$3x-2y+z=2;\quad 5x-4y+z=4.$$

What is the distance of each of these planes from the origin? What are the direction cosines of the perpendiculars to each? Which of the eight octants does each truncate?

3. Find the intercepts on the axes of $3x-2y+4z-12=0$; of $6x-4y-3z+24=0$; of $5x+7y+5z+35=0$. Which of the eight octants does each truncate? Reduce each equation to the symmetrical form.

4. What is the equation of the plane at the distance 7 from the origin, and perpendicular to the line whose direction cosines are proportional to $2, -3$, and $\sqrt{3}$?

5. What is the equation of the plane whose intercepts on the axes are respectively $4, -3, -7$? $-1, -2, -5$? $\frac{1}{2}, -\frac{2}{3}, \frac{1}{6}$?

6. Find the equation of the plane passing through the points $(1, 2, 3)$, $(0, 4, -1)$, and $(1, -1, 0)$.

7. Find the angle between the planes

$$2x+z-y=3,$$
$$z+x+2y=5.$$

8. Find the angle between the planes

$$3z+5x-7y=-1,$$
$$3z-2x-y=0.$$

9. Find the angle that the plane $Ax+By+Cz=D$ makes with each of the coördinate planes.

10. Find the distance from $(2, -3, 0)$ to the plane

$$\sqrt{3}z+2x-3y=4.$$

11. Show that the two points $(1, -1, 3)$ and $(3, 3, 3)$ are on opposite sides of, and equidistant from, the plane

$$5x+2y-7z+9=0.$$

12. If, in Fig. 92, $OM = a$, $OR = b$, $OS = c$, find the equation of the plane through the points M, P, R. Find the length of the perpendicular from S upon this plane.

13. Prove that the plane

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

passes through the point (x_1, y_1, z_1), and is parallel to the plane $Ax + By + Cz = D$.

14. Find the equation of the plane passing through the point $(3, 4, -1)$, and parallel to the plane $2x + 4y - z = 2$.

15. What three equations must be satisfied in order that the plane $Ax + By + Cz = D$ may pass through the two points (x_1, y_1, z_1), (x_2, y_2, z_2), and be perpendicular to the plane

$$A'x + B'y + C'z = D'?$$

16. Find the equation of the plane passing through the points $(1, 1, 1)$, $(2, 0, -1)$, and perpendicular to the plane $x + y - z = 3$.

17. What three equations must be satisfied in order that the plane $Ax + By + Cz = D$ may pass through the three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)?

18. Find the equation of the plane that passes through the points $(1, 2, 3)$, $(3, 2, 1)$, $(2, 3, 1)$, and find the distance of this plane from the origin.

19. Find the equation of the plane through $(2, 3, -1)$ parallel to the plane $3x - 4y + 7z = 0$.

20. Find the equation of the plane that passes through the point $(1, 2, 3)$, and is perpendicular to each of the planes $x + 2z = 1$, $y + 5z = 1$.

CHAPTER III.

THE STRAIGHT LINE.

226. *To find the equations of a straight line.*

The coördinates of any point on the line of intersection of two planes will satisfy the equation of each of these planes. Hence, any two simultaneous equations of the first degree in x, y, and z represent some straight line. Of the

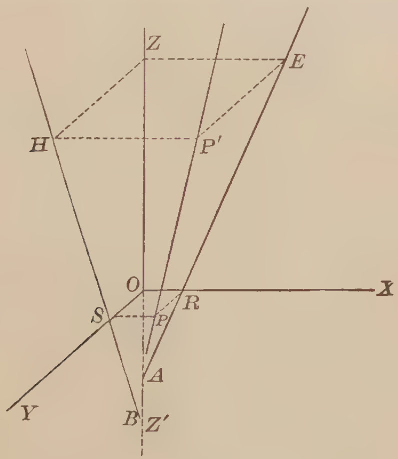


Fig. 99.

indefinite number of pairs of planes that intersect in, and therefore determine, a straight line, the equations of its projecting planes on the coördinate planes are the simplest, and two of them are taken as the equations of the line. Thus, let $PP'ER$ and $PP'HS$ be the projecting planes of any

straight line PP' on the coördinate planes xz and yz, respectively; and let the equations of these projecting planes be

$$\begin{cases} x = mz + p, & (1) \\ y = nz + q; & (2) \end{cases}$$

then are (1) and (2) the equations of the line PP'.

COR. 1. Let RE and SH be the *projections* of the line PP' on the planes xz and yz, respectively. Since the line RE lies in the plane $PP'RE$, equation (1) expresses the relation between the coördinates x, z of every point in RE; hence, (1) is the equation of RE referred to the axes ZZ' and OX. For like reason, (2) is the equation of the projection SH referred to the axes ZZ' and OY.

Hence, $m = \tan ZAE = \text{slope } RE$;
$p = OR = \text{intercept of } RE \text{ on } OX$;
$n = \tan ZBH = \text{slope } SH$;
$q = OS = \text{intercept of } SH \text{ on } OY$.

REM. The locus of (1) in space is the plane $PP'ER$, while its locus in the plane xz is the line RE. Similarly, the locus of (2) in space is the plane $PP'HS$, while its plane locus is SH. The locus in space of (1) and (2), considered as simultaneous, is the line PP'.

COR. 2. Eliminating z between (1) and (2), we obtain

$$y = \frac{n}{m}x + \left(q - \frac{np}{m}\right),$$

whose locus in space is the projecting plane of PP' on the plane xy, and whose locus in the plane xy is the projection of PP' on that plane.

COR. 3. Making $z = 0$ in equations (1) and (2), we obtain

$$x = p, \quad y = q;$$

hence, the line PP' pierces the plane xy in the point $(p, q, 0)$. This is evident also from the figure.

In like manner, we find that the line pierces the planes xz and yz respectively in the points

$$\left(\frac{np-mq}{n}, 0, -\frac{q}{n}\right), \quad \left(0, \frac{mq-np}{m}, -\frac{p}{m}\right).$$

227. *To find the symmetrical equations of a right line.*

Let α, β, γ be the direction angles of any right line, (x_1, y_1, z_1) some fixed point in it, and (x, y, z) any other point of the line. Let r denote the distance between these two points. Then by § 218, Cor. 2, we have

$$x - x_1 = r\cos\alpha, \quad y - y_1 = r\cos\beta, \quad z - z_1 = r\cos\gamma. \quad (1)$$

Whence, $$\frac{x-x_1}{\cos\alpha} = \frac{y-y_1}{\cos\beta} = \frac{z-z_1}{\cos\gamma}, \quad [63]$$

which are the symmetrical equations of a right line passing through the point (x_1, y_1, z_1).

Cor. If [63] passes through a second point (x_2, y_2, z_2), its coördinates must satisfy [63]; hence, we have

$$\frac{x_2-x_1}{\cos\alpha} = \frac{y_2-y_1}{\cos\beta} = \frac{z_2-z_1}{\cos\gamma}. \quad (2)$$

Dividing each member of [63] by the corresponding member of (2), we obtain

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}, \quad [64]$$

which are the equations of a right line through the two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

228. If we divide the denominators in any equations of the form

$$\frac{x-x_1}{L} = \frac{y-y_1}{M} = \frac{z-z_1}{N} \quad (1)$$

by $\sqrt{L^2+M^2+N^2}$, the denominators will then be the direction cosines of some line (§ 213, Cor.), and the equations will be in the form of [63].

Hence, to reduce equations in the form of (1) to the symmetrical form, *divide each denominator by the square root of the sum of the squares of the denominators.*

COR. The locus of equations (1) is the line through (x_1, y_1, z_1) parallel to the radius vector of the point (L, M, N).

229. *To find the angle between the lines*

$$\frac{x-x_1}{L}=\frac{y-y_1}{M}=\frac{z-z_1}{N},$$

and
$$\frac{x-x_2}{L'}=\frac{y-y_2}{M'}=\frac{z-z_2}{N'}.$$

By § 228 the direction cosines of these lines are respectively

$$\frac{L}{\sqrt{L^2+M^2+N^2}},\ \frac{M}{\sqrt{L^2+M^2+N^2}},\ \frac{N}{\sqrt{L^2+M^2+N^2}};$$

$$\frac{L'}{\sqrt{L'^2+M'^2+N'^2}},\ \frac{M'}{\sqrt{L'^2+M'^2+N'^2}},\ \frac{N'}{\sqrt{L'^2+M'^2+N'^2}}.$$

Substituting these values in [57], we obtain

$$\cos\theta=\frac{LL'+MM'+NN'}{\sqrt{L^2+M^2+N^2}\ \sqrt{L'^2+M'^2+N'^2}}. \qquad [65]$$

COR. 1. If the lines are parallel, $\frac{L}{L'}=\frac{M}{M'}=\frac{N}{N'}$, and conversely.

COR. 2. If the lines are perpendicular,

$$LL'+MM'+NN'=0,$$

and conversely.

230. *To find the inclination of the line*

$$\frac{x-x_1}{L}=\frac{y-y_1}{M}=\frac{z-z_1}{N} \qquad (1)$$

to the plane $Ax+By+Cz=D.$ (2)

The equation of the perpendicular from (x_1, y_1, z_1) to the plane is

$$\frac{x-x_1}{A}=\frac{y-y_1}{B}=\frac{z-z_1}{C}. \qquad (3)$$

Now, the inclination of line (1) to plane (2) is evidently the complement of the angle between the lines (1) and (3). Denote this inclination by v; then $\sin v = \cos\theta$, θ being the angle between the lines (1) and (3). Hence,

$$\sin v = \frac{AL+BM+CN}{\sqrt{A^2+B^2+C^2}\,\sqrt{L^2+M^2+N^2}}. \qquad [66]$$

Cor. 1. If the line is parallel to the plane, $\sin v = 0$, and, therefore, $AL+BM+CN=0$, and conversely.

Cor. 2. If the line is perpendicular to the plane, $\sin v = 1$, and, therefore, $\frac{L}{A}=\frac{M}{B}=\frac{N}{C}$, and conversely.

Cor. 3. If line (1) lies in plane (2), then

$$AL+BM+CN=0,$$

and

$$Ax_1+By_1+Cz_1=D,$$

and conversely.

Exercise 45.

1. Determine the position, direction cosines, and direction angles of the intersection of the planes $x+y-z+1=0$, and $4x+y-2z+2=0$.

Eliminating successively y and z between the equations, we obtain $3x-z+1=0$ and $2x-y=0$; or $\frac{x}{1}=\frac{y}{2}=\frac{z-1}{3}$.

From the last form we know that the line passes through the point (0, 0, 1), and is parallel to the radius vector of the point (1, 2, 3). The direction cosines are found by dividing the denominators 1, 2, 3, by $\sqrt{14}$; and the direction angles are found from their cosines.

2. Determine the position and direction cosines of the intersection of $x-2y=5$ and $3x+y-7z=0$.

Here $\frac{x-5}{2}=\frac{y}{1}=\frac{z-\frac{15}{7}}{1}$, whence the line passes through the point $(5, 0, \frac{15}{7})$, and is parallel to the radius vector of $(2, 1, 1)$.

3. Determine the position of the line

$$5x-4y=1,\ 3y-5z=2.$$

4. What is the position of the line $x=3$, $y=4$? Of the line $y=4$, $z=-5$? Of the line $x=-2$, $z=3$?

5. Find the equations of the right line passing through the points $(1, 2, 3)$, $(3, 4, 1)$.

6. Find the points in which the line of Example 5 pierces the coördinate planes.

7. Two of the projecting planes of a line are $x+y=4$ and $2x-5z=-2$; find the third.

8. A line passes through $(2, 1, -1)$ and $(-3, -1, 1)$; find the equations of its projections on the coördinate planes.

9. Show that the lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{-1}=\frac{z}{1}$ are at right angles.

10. Show that the line $4x=3y=-z$ is perpendicular to the line $3x=-y=-4z$.

11. Find the angle between the lines

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{0} \text{ and } \frac{x}{3}=\frac{y}{-4}=\frac{z}{5}.$$

12. Find the angle between the right lines

$$y=5x+3,\ z=3x+5,$$

and

$$y=2x,\ z=x+1.$$

13. Find the angle between the lines

$$y=2x+2,\ z=2x+1,$$

and

$$y=4x+1,\ z=x+5.$$

14. Show that the lines

$$3x + 2y + z - 5 = 0, \ x + y - 2z - 3 = 0,$$

and $$8x - 4y - 4z = 0, \ 7x + 10y - 8z = 0$$

are at right angles.

15. Find the equations of the line through $(-2, 3, -1)$ parallel to the line $y = -2x + 1$, $z = 3x - 4$.

16. Find the equations of the line through $(3, -7, -5)$, its direction cosines being proportional to $-3, 5, -6$.

17. Find the equations of the line through $(2, -4, -6)$ perpendicular to the plane $3x - 6y + 2z = 4$.

18. Find the inclination of the line

$$\frac{x-4}{3} = \frac{y+2}{-2} = \frac{z-5}{-4}$$

to the plane $2x - 4y + 3z = 1$.

19. Reduce the equations, $x = mz + p$, $y = nz + q$, to the symmetrical form, and thus find the direction cosines in terms of m and n.

20. Show that the formula for the angle included between the lines

$$x = mz + p, \quad y = nz + q,$$

and $$x = m'x + p', \ y = n'z + q'$$

is $$\cos\theta = \frac{mm' + nn' + 1}{\sqrt{m^2 + n^2 + 1}\ \sqrt{m'^2 + n'^2 + 1}}.$$

21. Prove that the lines in Example 20 are perpendicular if $mm' + nn' + 1 = 0$, and conversely. Prove that they are parallel if $m = m'$, and $n = n'$, and conversely.

22. Prove that two lines are parallel if their projections are parallel, and conversely.

SUPPLEMENTARY PROPOSITIONS.

231. The **Traces** of a plane are its lines of intersection with the coördinate planes. Thus AB, BC, CA (Fig. 98) are the traces of the plane ABC.

232. *To find the equations of the traces of the plane*

$$Ax + By + Cz = D. \tag{1}$$

For every point in the plane xy, $z = 0$; hence, putting $z = 0$ in (1), we obtain

$$Ax + By = D, \tag{2}$$

which is the equation of the trace AB (Fig. 98) on the plane xy. For like reason,

$$By + Cz = D \tag{3}$$

and

$$Ax + Cz = D \tag{4}$$

are the equations of the traces BC and CA on the planes yz and xz, respectively.

Cor. The perpendicular from the origin to (1) is

$$\frac{x}{A} = \frac{y}{B} = \frac{z}{C},$$

and its projections on the coördinate planes are

$$Bx = Ay,\ Cy = Bz,\ Cx = Az. \tag{5}$$

By comparing coefficients, we see that lines (5) are perpendicular to (2), (3), and (4), respectively. Hence,

If a line in space is perpendicular to a plane, its projections are perpendicular to the traces of the plane.

233. *To find the condition that the right lines*

$$x = mz + p, \qquad y = nz + q,$$

and

$$x = m'z + p', \qquad y = n'z + q',$$

may intersect, and to find their points of intersection.

Equating the two values of x and y, we have

$$z = \frac{p' - p}{m - m'},\ z = \frac{q' - q}{n - n'}.$$

If the two lines intersect, these two values of z must be equal; hence, $\frac{p' - p}{m - m'} = \frac{q' - q}{n - n'}$ is the equation of condition that the two given lines in space intersect.

When this condition is fulfilled, the values of x and y may be found by substituting either value of z in the equations of either line.

234. To pass a plane through the point (x_2, y_2, z_2) and the right line

$$\frac{x - x_1}{L} = \frac{y - y_1}{M} = \frac{z - z_1}{N}. \tag{a}$$

If the plane $Ax + By + Cz = D$ (1)

passes through the point (x_2, y_2, z_2), we have

$$Ax_2 + By_2 + Cz_2 = D; \tag{2}$$

and if line (a) lies in plane (1), we have

$$Ax_1 + By_1 + Cz_1 = D, \tag{3}$$

and $$AL + BM + CN = 0. \tag{4}$$

The equation of the required plane is found by eliminating A, B, C, D from (1), (2), (3), and (4).

To simplify the process of elimination, (1) might be written in the form $A'x + B'y + C'z = 1$, but the solution would be less general, as it would not embrace the case when $D = 0$.

235. From the forms $x = mz + p$, $y = nz + q$, show that the equations of a line passing through (x_1, y_1, z_1) are

$$\begin{cases} x - x_1 = m(z - z_1), \\ y - y_1 = n(z - z_1). \end{cases}$$

CHAPTER IV.

SURFACES OF REVOLUTION.

236. It has been shown that a single equation of the first degree between three variables represents a plane surface, and that two such equations in general represent a right line. It is evident, moreover, that in general three such equations determine a point common to their loci. Thus, if in Fig. 92, $OM = a$, $OR = b$, and $OS = c$, then the equations $x = a$, $y = b$, $z = c$ determine the point P, and are called its equations. We proceed to show that,

In general, any single equation of the form $f(x, y, z) = 0$ represents a surface of some kind: two such equations represent a curve, and three determine one or more points.

(i) Let two of the variables be absent; for example, let the equations be $f(x) = 0$. Now $f(x) = 0$ may be written in the form

$$(x - a_1)(x - a_2)(x - a_3) \ldots\ldots (x - a_n) = 0, \tag{1}$$

in which a_1, a_2, a_3, ... a_n are the n roots of $f(x) = 0$. The locus of (1) is evidently the n parallel planes $x = a_1$, $x = a_2$, ..., $x = a_n$. Similarly, the equations $f(y) = 0$, $f(z) = 0$ represent planes perpendicular to the axes of y and z, respectively.

(ii) Let one of the variables be absent; for example, let the equation be $f(x, y) = 0$. The locus of $f(x, y) = 0$ in the plane xy is some plane curve. Through P, any point in this curve, conceive a line parallel to the axis of z; then the coördinates x, y of all points in this line will equal those of P, and hence satisfy the equation $f(x, y) = 0$. Hence, the

locus in space of $f(x, y) = 0$ is the surface generated by a right line which is always parallel to the axis of z, and which moves along the plane locus of $f(x, y) = 0$.

That is, *the locus in space of $f(x, y) = 0$ is a cylindrical surface whose elements are parallel to the axis of z, and whose directrix is the plane locus of $f(x, y) = 0$.*

Similarly, the equations $f(x, z) = 0$ and $f(y, z) = 0$ represent cylindrical surfaces whose elements are parallel to the axes of y and x, respectively.

(iii) Let the equation be $f(x, y, z) = 0$. If in this equation we put $x = a$ and $y = b$, the roots of the resulting equation in z will give the points in the locus that lie on the line through $(a, b, 0)$ parallel to the axis of z. But as the number of these roots is finite, the number of points of the locus on this line is finite. Hence, the locus which embraces all such points for different values of a and b must be a surface and not a solid.

(iv) Two equations considered as simultaneous are satisfied by the coördinates of all the points of intersection of their loci; that is, they represent the curve of intersection of two surfaces.

(v) Three independent simultaneous equations are satisfied only by the coördinates of the points in which the curve represented by two of them cuts the surface represented by the third; hence they determine these points.

Cor. 1. From (ii) it follows that $x^2 + y^2 = r^2$ is the equation of a cylinder whose axis is the axis of z, and whose radius is r. Also,

$$y^2 = 4px, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ and } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

are equations of cylindrical surfaces whose elements are parallel to the axis of z, and whose directrices are respectively the parabola, the ellipse, and the hyperbola.

COR. 2. If $F(x, y) = 0$ is the equation obtained by eliminating z between the two equations $f(x, y, z) = 0$ and $f_1(x, y, z) = 0$, then the locus in space of $F(x, y) = 0$ is the projecting cylinder on the plane xy of the curve represented by the two equations. The *plane* locus of $F(x, y) = 0$ is the projection of this curve on the plane xy. If the curve is parallel to the plane of projection, the curve and its projection are equal.

The equation obtained by eliminating x or y between the two equations has evidently a like interpretation.

237. The **Traces of a Surface** are its intersections with the coördinate planes.

If $f(x, y, 0) = 0$ denotes the equation obtained by making $z = 0$ in $f(x, y, z) = 0$, then the plane locus of $f(x, y, 0) = 0$ is evidently the *trace* of the *surface* $f(x, y, z) = 0$ on the plane xy.

SURFACES OF REVOLUTION.

238. A **Surface of Revolution** is a surface that may be generated by a curve revolving about a fixed straight line as an axis. The revolving curve is called the **Generatrix**; and the fixed right line, the **Axis of Revolution,** or simply the **Axis.** A section of the surface made by a plane passing through the axis is called a **Meridian Section.** From these definitions, it follows that

(i) Every section made by a plane perpendicular to the axis is a circle, whose centre is in the axis.

(ii) Any meridian section is equal to the generatrix.

239. *To find the general equation of a surface of revolution.*

Let the axis of z be the axis of revolution, and let P be any point in the meridian section made by the plane xz. Let PHR be a section through P perpendicular to the axis of z, and denote the radius CH, or CP, of this circular section by r.

Now for all points in this circular section, we have $x^2 + y^2 = r^2$, and $z = MP$. The value of r^2, in terms of z, is obtained by substituting r for x in the equation of the meridian section made by the plane zx. Denoting this value of r^2 by $f(z)$, and equating the two values of r^2, we have

$$x^2 + y^2 = f(z), \qquad [67]$$

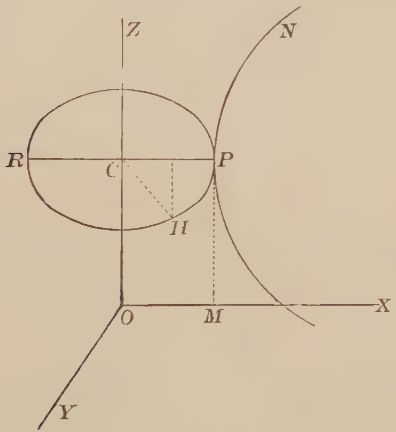


Fig. 100.

which expresses the relation between the coördinates x, y, z of all points in the section PHR. But as P is any point in the meridian section NP, [67] is the general equation of a surface of revolution whose axis is the axis of z.

240. Paraboloid of Revolution. A *Paraboloid of Revolution* is a surface that may be generated by a parabola revolving about its axis.

In this case the equation of the meridian section in the plane zx is

$$x^2 = 4pz;$$

hence,

$$r^2 = 4pz = f(z).$$

Substituting in [67], we obtain

$$\boldsymbol{x^2 + y^2 = 4pz,} \qquad [68]$$

which is *the equation of the paraboloid of revolution.*

If in [68] we put $x = m$, we obtain

$$y^2 = 4pz - m^2, \qquad (1)$$

which is the equation of the projection, on the plane yz, of the section of the paraboloid made by a plane parallel to the plane yz, and at a distance from it equal to m. Now the plane locus of (1), for all values of m, is a parabola; hence, every plane section of the paraboloid parallel to the plane yz is a parabola. If in [68] we put $y = n$, we obtain

$$x^2 = 4pz - n^2. \qquad (2)$$

From (2) we learn that all plane sections parallel to the plane xz are also parabolas. From definition, we know that all plane sections parallel to the plane xy are circles.

241. Ellipsoid of Revolution. An *Ellipsoid of Revolution*, or *Spheroid*, is a surface that may be generated by an ellipse revolving about one of its axes. It is called **Oblate** when the revolution is about the minor axis; and **Prolate** when about the major axis.

(i) When the revolution is about the minor axis, the equation of the meridian section in the plane xz is

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1; \text{ hence, } r^2 = a^2\left(1 - \frac{z^2}{b^2}\right) = f(z).$$

Substituting in [67], and reducing, we obtain

$$\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1,} \qquad [69]$$

which is *the equation of the oblate spheroid.*

Cor. 1. If $a = b$, [69] becomes

$$x^2 + y^2 + z^2 = a^2, \qquad (1)$$

which is the equation of a sphere whose radius is a.

COR. 2. If in [69] we put $x = m$, we obtain

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 - \frac{m^2}{a^2}. \tag{2}$$

Since (2) represents an ellipse, a point, or no locus in the plane yz, according as $m^2 <, =,$ or $> a^2$, the surface lies between the two tangent planes $x = a$ and $x = -a$, and all plane sections parallel to the plane yz are ellipses.

If in [69] we put $y = n$, we obtain

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 - \frac{n^2}{a^2}. \tag{3}$$

Since (3) represents an ellipse, a point, or no locus in the plane zx, according as $n^2 <, =,$ or $> a^2$, the surface lies between the two tangent planes $y = a$ and $y = -a$, and all plane sections parallel to the plane zx are ellipses.

If in [69] we put $z = q$, we obtain

$$x^2 + y^2 = a^2\left(1 - \frac{q^2}{b^2}\right). \tag{4}$$

Equation (4) represents a circle, a point, or no locus in the plane xy, according as $q^2 <, =,$ or $> b^2$. Hence, the surface lies between the tangent planes $z = b$ and $z = -b$, and all plane sections parallel to the plane xy are circles.

(ii) When the revolution is about the major axis, the equation of the meridian section in the plane xz is

$$\frac{z^2}{a^2} + \frac{x^2}{b^2} = 1; \text{ hence, } r^2 = b^2\left(1 - \frac{z^2}{a^2}\right) = f(z).$$

Substituting in [67], we obtain

$$\frac{\boldsymbol{x^2}}{\boldsymbol{b^2}} + \frac{\boldsymbol{y^2}}{\boldsymbol{b^2}} + \frac{\boldsymbol{z^2}}{\boldsymbol{a^2}} = \mathbf{1}, \tag*{[70]}$$

which is *the equation of the prolate spheroid.*

If in [69] we interchange a and b, we obtain [70]. Hence, we interchange a and b in the discussion of [69].

242. Hyperboloid of Revolution. An *Hyperboloid of Revolution* is a surface that may be generated by an hyperbola revolving about one of its axes. It consists of one or two nappes, or sheets, according as the hyperbola revolves about its conjugate or transverse axis.

(i) If in [69] we substitute $-b^2$ for b^2, we obtain

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{b^2}=1, \tag{71}$$

which is *the equation of the hyperboloid of one nappe.*

If in [71] we put $x=m$, we have

$$\frac{y^2}{a^2}-\frac{z^2}{b^2}=1-\frac{m^2}{a^2}, \tag{1}$$

whose plane locus is an hyperbola for all values of m. Hence, all plane sections parallel to the plane yz are hyperbolas. The transverse axis of any one of these hyperbolas is evidently parallel to the axis of y or z, according as $m^2 <$ or $> a^2$. If $m^2=a^2$ (1) becomes

$$y=\pm\frac{a}{b}z.$$

Hence, the sections of [71] made by the planes $x=\pm a$ are each two intersecting right lines.

If in [71] we put $y=n$, we have

$$\frac{x^2}{a^2}-\frac{z^2}{b^2}=1-\frac{n^2}{a^2}. \tag{2}$$

Hence, all plane sections of [71] parallel to the plane xz are hyperbolas, whose transverse axes are parallel to the axis of x or z, according as $n^2 <$ or $> a^2$; and the sections made by the planes $y=\pm a$ are each two intersecting right lines.

If in [71] we put $z=q$, we obtain

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1+\frac{q^2}{b^2},$$

whose plane locus is a circle for all values of q. This circle is smallest when $q=0$. This smallest circle, which is the trace of the hyperboloid on the plane xy, is called the **Circle of the Gorge.**

(ii) If in [70] we substitute $-b^2$ for b^2, we obtain

$$\frac{x^2}{b^2}+\frac{y^2}{b^2}-\frac{z^2}{a^2}=-1, \qquad [72]$$

which is *the equation of the hyperboloid of two nappes.*

The discussion of [72] for parallel plane sections is left as an exercise for the student.

243. The **Centre** of a surface is a point that bisects all chords passing through it.

Central Surfaces are such as have a centre.

The ellipsoids and hyperboloids of revolution are central surfaces. For, from their equations, it is evident that, if (x', y', z') is a point in any one of these surfaces, $(-x', -y', -z')$ is also a point in the same surface. But the chord joining these two points is bisected by the origin, which is, therefore, the centre of the surface.

244. Cone of Revolution. A *Cone of Revolution* is a surface that may be generated by a right line revolving about an axis which it intersects.

Here the equation of the meridian section in plane xz is

$$z = mx + c;$$

therefore, $$r^2=\left(\frac{z-c}{m}\right)^2=f(z).$$

Whence, $$m^2(x^2+y^2)=(z-c)^2 \qquad [73]$$

is *the equation of the cone of revolution.*

In this equation c is the distance of the vertex from the origin and $m = \tan XDB$.

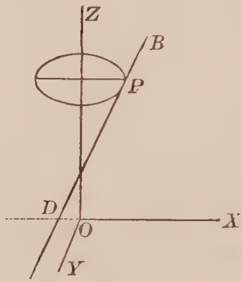


Fig. 101.

If $c=0$, [73] becomes

$$m^2(x^2+y^2)=z^2. \tag{1}$$

From (1) it is evident that the cone is a central surface.

If in (1) we put $y=n$, we obtain

$$\frac{z^2}{n^2m^2}-\frac{x^2}{n^2}=1,$$

whose plane locus is an hyperbola for all values of n. Hence, all plane sections of the cone parallel to the plane zx are hyperbolas whose transverse axes are parallel to the axis of the cone. In like manner, we find that all plane sections parallel to the plane yz are hyperbolas. If $y=0$, $z=\pm mx$, whose locus is two intersecting right lines. Hence, *any plane section of a cone parallel to its axis is an hyperbola, and any section containing the axis is two intersecting right lines.*

A **Conic Section** is the section of a cone made by a plane.

245. To determine the nature of any conic section that is not parallel to the axis of the cone, we find the equation of any such section referred to axes in its own plane.

Let HPN be any section of the cone $VDY'N$ passing through the axis of y; then this section will be perpendicular to the plane xz. The cone, and therefore the section,

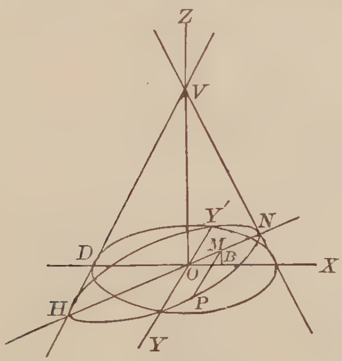


Fig. 102.

is symmetrical with respect to the plane xz. Refer this section to ON and OY as the axes of x and y respectively. Let (x, y, z) be the point P referred to the coördinate planes, and (x', y') be P referred to ON and OY. Let $XON = \phi$ and $ODV = \theta$. Draw PM perpendicular to ON; then it will be perpendicular to the plane xz, and we have

$$y = y',\ OB = OM\cos\phi, \text{ or } x = x'\cos\phi;$$

$$BM = OM\sin\phi, \text{ or } z = x'\sin\phi.$$

Substituting these values of x, y, z in [73], we obtain

$$\tan^2\theta\,(x'^2\cos^2\phi + y'^2) = (x'\sin\phi - c)^2.$$

Omitting accents and performing indicated operations, we have

$$y^2\tan^2\theta + x^2(\cos^2\phi\tan^2\theta - \sin^2\phi) + 2cx\sin\phi - c^2 = 0.$$

Substituting $\cos^2\phi\tan^2\phi$ for $\sin^2\phi$. we obtain

$$\mathbf{y^2\tan^2\theta + x^2\cos^2\phi\,(\tan^2\theta - \tan^2\phi) + 2cx\sin\phi - c^2 = 0,} \quad [74]$$

which is the equation of the conic NPH referred to ON and OY as axes.

By giving to c all values between 0 and ∞, and to ϕ all values between 0° and 90°, equation [74] is made to represent any section of a cone except those parallel to its axis, which have already been considered.

Discussion of equation [74].

Here $\Sigma = 4\cos^2\phi\tan^2\theta(\tan^2\theta - \tan^2\phi)$,

$$\Delta = 4c^2[\cos^2\phi\tan^2\theta(\tan^2\theta - \tan^2\phi) + \tan^2\theta\sin^2\phi].$$

(i) *First suppose c not equal to zero.*

Let $\phi < \theta$; then $\tan^2\phi < \tan^2\theta$, Σ is positive, and Δ is not zero; hence the section is an ellipse.

Let $\phi = \theta$; then $\tan^2\phi = \tan^2\theta$, $\Sigma = 0$, and Δ is not zero; hence the section is a parabola.

Let $\phi > \theta$; then is $\tan^2\phi > \tan^2\theta$, Σ is negative, and Δ is not zero; hence, the section is an hyperbola.

Hence, when the cutting plane does not pass through the vertex of the cone, *the section is an ellipse, a parabola, or an hyperbola, according as the angle which the cutting plane makes with the base of the cone is less than, equal to, or greater than that made by an element.*

(ii) If $c = 0$, $\Delta = 0$; hence, when the cutting plane passes through the vertex, the elliptical section reduces to a point, the parabolic to a straight line, and the hyperbolic to two intersecting right lines.

If $\phi = 0$, the cutting plane is perpendicular to the axis of the cone, and equation [74] becomes

$$y^2 + x^2 = c^2\cot^2\theta,$$

whose locus is a circle.

If $c = \infty$, the cone becomes a cylinder, and the section made by a plane parallel to an element is two parallel lines or a single right line.

Exercise 46.

1. What is the locus in space of $x^3 + 3x^2 - 6x - 8 = 0$? of $y^3 - 2y^2 - 5y + 6 = 0$? of $z^2 + mz = 0$?

2. What is the locus in space of $y^2 = 8x$? of $4x^2 + 9y^2 = 36$? of $9z^2 - 16y^2 = 144$? of $(2a - z)(y^2 - b^2) = 0$? of $z^2 + x^2 = r^2$?

3. Find the equations of the projecting cylinders of the curve $x^2 + 3y^2 - 2z^2 = 8$, $x^2 + 2y^2 + 3z^2 = 16$?

4. Find the equations of the projections of the curves

$$x^2 + y^2 + 2z^2 = 16,\ 9(x^2 + y^2) + 4z^2 = 36.$$

5. Find the semi-axes and eccentricity of the ellipse

$$4x^2 + 9y^2 + 4z^2 = 37,\ z = \tfrac{1}{2}.$$

6. Find the nature of the curves

$$x^2 + y^2 + 4z^2 = 25,\ 7(x^2 + y^2) - 4z^2 = 79.$$

7. Find the traces of the surface $2x^2 + 5y^2 - 7z^2 = 9$; of the surface $x^2 + 3y^2 = 8z$.

8. Find the equation of the surface of revolution whose axis is the axis of z, and one of whose traces is $z = \pm 3x + 5$; find its trace on the plane xy.

9. Find the equation of a cone of revolution one of whose traces is $x^2 + y^2 = 9$, and whose vertex is $(0, 0, 5)$.

10. Find the equation of the paraboloid of revolution one of whose traces is $2x^2 = 3z + 5$.

11. Find the equation of the paraboloid of revolution one of whose traces is $y^2 = 8x$.

12. Find the equation of the cone of revolution whose axis is the axis of z, and one of whose traces is $2y = \pm z + 6$; find its vertex.

13. Find the equation of the surface of revolution whose axis is the axis of z, and one of whose traces is $9x^2+4z^2=36$.

14. Find the equation of the surface of revolution whose axis is the axis of z, and one of whose traces is $16y^2+9z^2=144$.

15. Find the equation of the surface of revolution whose axis is the axis of z, and one of whose traces is $9z^2-4y^2=-36$.

16. Find the equation of the surface of revolution whose axis is the axis of z, and one of whose traces is $z^2x=1$; also when one trace is $z^3=2y^2$.

17. Each element of a cone makes an angle of 45° with its axis; find the semi-axes of the section made by a plane cutting the axis 5 below the vertex and at an angle of 60°.

SUPPLEMENTARY PROPOSITIONS.

246. *To find the general equation of the sphere.*

Let r denote the radius of any sphere, (a, b, c) its centre, and (x, y, z) any point on its surface. Then, since r is the distance between the points (a, b, c) and (x, y, z), we have

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2, \tag{75}$$

or $$x^2+y^2+z^2-2ax-2by-2cz=r^2-a^2-b^2-c^2, \tag{1}$$

which is *the general rectangular equation of the sphere.*

If the origin is at the centre, then $a=b=c=0$, and [75] becomes

$$x^2+y^2+z^2=r^2. \tag{2}$$

From (1) it follows that any equation of the form

$$x^2+y^2+z^2+Gx+Hy+Iz=K \tag{3}$$

is the equation of a sphere.

Any equation of the form of (3) can readily be reduced to the form of [75], from which the centre and radius of its locus become known.

Since (3) or [75] contains four arbitrary constants, a sphere may in general be made to pass through any four given points.

247. *The intersection of two spheres is a circle.*

Let the equations of the two spheres be

$$x^2+y^2+z^2+Gx+Hy+Iz=K, \tag{1}$$

and

$$x^2+y^2+z^2+G'x+H'y+I'z=K'. \tag{2}$$

Subtracting (2) from (1), we obtain

$$(G-G')x+(H-H')y+(I-I')z=K-K'. \tag{3}$$

Hence, the intersection of the spheres (1) and (2) lies in the plane (3), and is the same as the intersection of (1) and (3). But the plane section of a sphere is a circle. Hence, the intersection of the two spheres is a circle.

248. *To find the equation of the tangent plane to a sphere at a given point.*

Let the given point be (x_1, y_1, z_1); then the equation of the radius to this point, that is, of the line passing through (a, b, c) and (x_1, y_1, z_1), is

$$\frac{x-x_1}{a-x_1}=\frac{y-y_1}{b-y_1}=\frac{z-z_1}{c-z_1}. \tag{1}$$

Now the tangent plane is perpendicular to (1) at the point (x_1, y_1, z_1); but the equation of the plane through (x_1, y_1, z_1) perpendicular to (1) is

$$(a-x_1)(x-x_1)+(b-y_1)(y-y_1)+(c-z_1)(z-z_1)=0, \tag{2}$$

which is, therefore, *the equation of the tangent plane.*

If the origin is at the centre, $a=b=c=0$, and (2) becomes

$$xx_1+yy_1+zz_1=r^2.$$

Transformation of Coördinates.

249. *To change the origin of coördinates without changing the direction of the axes.*

Let (m, n, q) be the new origin referred to the old axes. Let x, y, z be the old, and x', y', z' the new coördinates of any point P; then, evidently, we have

$$x = m + x',\ y = n + y',\ z = q + z'.$$

Hence, to find the equation of a locus referred to new parallel axes whose origin is (m, n, q), *substitute* $m + x$, $n + y$, *and* $q + z$, *for* x, y, *and* z, *respectively.*

250. *To change the direction of the axes without changing the origin.*

Let $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$; $\alpha_3, \beta_3, \gamma_3$ be, respectively, the direction angles of the new axes OX', OY', OZ' referred to the

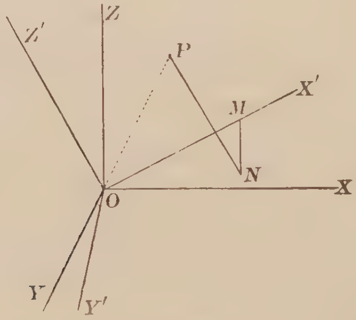


Fig. 103.

old axes OX, OY, OZ. Let x, y, z be the old, and x', y', z' the new coördinates of any point P. Draw PN perpendicular to the plane $X'OY'$, and NM perpendicular to OX'; then $OM = x'$, $MN = y'$, and $NP = z'$. Now the projection of

OP on OX ($=x$) is equal to the sum of the projections of OM, MN, and NP on the same line; hence,

$$x = x' \cos \alpha_1 + y' \cos \alpha_2 + z' \cos \alpha_3. \qquad (1)$$

In like manner, we obtain

$$y = x' \cos \beta_1 + y' \cos \beta_2 + z' \cos \beta_3, \qquad (2)$$

and
$$z = x' \cos \gamma_1 + y' \cos \gamma_2 + z' \cos \gamma_3. \qquad (3)$$

Hence, to change the direction of the axes without changing the origin, *substitute for x, y, and z their values as given in equations* (1), (2), *and* (3).

Since the values of x, y, z are each of the first degree in x', y', z', any transformation of coördinates cannot change the degree of an equation. (§ 91.)

251. Quadrics. The locus of an equation of the second degree that contains three variables is called a **Quadric.** Thus the general equation of a quadric is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + K = 0. \qquad (1)$$

Putting $z = q$ in (1), we obtain

$$Ax^2 + Dxy + By^2 + (Eq + G)\,x + (Fq + H)y + (Cq^2 + Iq + K) = 0. \qquad (2)$$

Since the locus of (2) in the plane xy is a conic, and since the coefficients A, D, B are the same for all values of q, all plane sections of the quadric (1), parallel to the plane xy, are similar conics. Now the axis of coördinates may be so changed that the new plane xy will be one of any system of parallel planes cutting the quadric. But, as this transformation does not change the degree of the equation, it follows that

All parallel plane sections of any quadric are similar conics.

252. By transformations of coördinates the general equation (1) of § 251 may be reduced to one of the two following simple forms: *

$$Px^2 + Qy^2 + Rz^2 = S. \qquad (1)$$

$$Px^2 + Qy^2 = Uz. \qquad (2)$$

Now whatever be the values or signs of P, Q, R, S, equation (1) evidently represents central quadrics. But the loci of (2) have no centre; for if they had, and the origin were changed to that centre, the first power of z would disappear from the equation. But no expression of the form $q + z$, when substituted for z, can cause z to disappear.

Hence, (2) represents non-central quadrics.

253. Central Quadrics. If no one of the coefficients P, Q, R, S is zero in (1) of § 252, we have

$$\frac{x^2}{S \div P} + \frac{y^2}{S \div Q} + \frac{z^2}{S \div R} = 1,$$

which can be written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad (A)$$

---

* By changing the direction of the axes the general equation can in all cases be reduced to the form

$$Px^2 + Qy^2 + Rz^2 + G'x + H'y + I'z - K = 0. \qquad (1)$$

This transformation is analogous to that in § 189.

(i) If no one of the three coefficients P, Q, R is zero, by a change of origin, as in § 188, we obtain

$$Px^2 + Qy^2 + Rz^2 = S. \qquad (2)$$

(ii) If any one of these coefficients is zero, for example R, by a change of origin, we obtain

$$Px^2 + Qy^2 = Uz. \qquad (3)$$

If two of these coefficients are zero, (1) can be reduced to a form embraced in (3) by first changing the origin and then the direction of the axes.

or $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \tag{B}$$

or $$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \tag{C}$$

according as $S\div P$, $S\div Q$, $S\div R$ are all positive, two positive and one negative, or one positive and two negative. [If all three are negative there is no real locus.]

If S is zero, we have

$$Px^2+Qy^2+Rz^2=0. \tag{D}$$

If P, Q, or R is zero in (1), its locus is a cylindrical surface by (ii) of § 236.

254. A discussion of (A) discovers the following properties of its locus:

(i) Its traces on each of the coördinate planes are ellipses.

(ii) All plane sections parallel to either coördinate plane are similar ellipses.

(iii) The quadric is included between the tangent planes

$$x=\pm a,\ y=\pm b,\ z=\pm c.$$

The quadric (A) is called an **Ellipsoid.** If $a=b$, the ellipsoid is the oblate or prolate spheroid, according as $a >$ or $< c$.

The ellipsoid may evidently be generated by a variable ellipse moving parallel to the plane xy with its centre in the axis of z, its axes being chords of the traces of the quadric on the planes yz and zx.

255. From a discussion of (B), we learn that:

(i) Its trace on the plane xy is an ellipse, while its traces on the planes yz and zx are hyperbolas, whose transverse axes lie on the axes of y and x respectively.

(ii) All plane sections parallel to the plane xy are ellipses, while all plane sections parallel to the plane yz and zx are

hyperbolas. The smallest elliptical section is the trace on the plane xy. The semi-axes of this ellipse are a and b.

The locus of (B) is called the **Hyperboloid of One Nappe.**

If $a = b$, the locus of (B) is an hyperboloid of revolution.

The hyperboloid can evidently be traced by a variable ellipse parallel to the plane xy, whose centre moves along the axis of z, and whose axes are the chords of the traces of the quadric on the planes yz and zx.

256. From a discussion of (C), we learn that:

(i) Its traces on the planes yx and zx are hyperbolas whose tranverse axes are on the axis of x.

(ii) The plane sections parallel to the plane zy are ellipses, and no portion of the quadric lies between the tangent planes $x = \pm a$.

(iii) The plane sections parallel to the planes yx and zx are hyperbolas whose transverse axes are parallel to the axis of x.

The locus of (C) is called the **Hyperboloid of Two Nappes.**

257. If the coefficients of (D) are all positive or all negative, its locus is the point (0, 0, 0). If two coefficients are negative and one positive, by dividing by -1, two become positive and one negative. Hence, we need discuss only the form represented by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \tag{D'}$$

from which we learn that:

(i) All plane sections parallel to the planes yz and zx are hyperbolas whose transverse axes are parallel to the axis of z.

(ii) All plane sections parallel to the plane xy are ellipses, the trace on this plane being a point.

(iii) The traces on the planes yz and zx are each two right lines intersecting at the origin.

(iv) All plane sections through the axis of z are two right lines intersecting at the origin.

For, denote any plane through the axis of z by

$$y = mx. \tag{1}$$

Eliminating y between (1) and (D'), we obtain

$$z = \pm \frac{cx}{ab}\sqrt{b^2 + a^2m^2}. \tag{2}$$

Now the intersections of (1) and (D′) are the same as the intersections of (1) and (2), which are evidently two right lines passing through the origin.

Hence, the locus of (D′) is a cone whose axis is the axis of z, and whose directrix is an ellipse. If $a = b$, it becomes a cone of revolution.

258. Non-Central Quadrics. If no one of the coefficients P, Q, U is zero in (2) of § 252, we have

$$\frac{x^2}{U \div P} + \frac{y^2}{U \div Q} = z,$$

which can be written in the form

$$\frac{x^2}{l} + \frac{y^2}{l'} = z, \tag{E}$$

or

$$\frac{x^2}{l} - \frac{y^2}{l'} = z, \tag{F}$$

according as P and Q have like or unlike signs.

A discussion of (E) discovers the following properties:

(i) Plane sections parallel to the plane xy are ellipses, and the surface lies above the tangent plane $z = 0$.

(ii) All plane sections parallel to the plane yz or zx are parabolas, and the traces on these planes are parabolas, having the axis of z as their common axis and their concavities upward.

By a discussion of (F) we learn that :

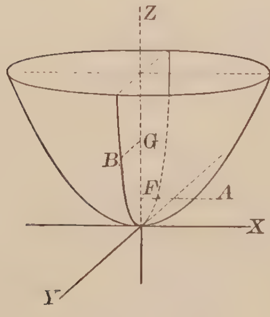
(i) The traces on the planes yz and zx are parabolas whose axes lie on the axis of z, and whose concavities are in opposite directions.

(ii) Plane sections parallel to the planes yz and zx are parabolas whose concavities are in opposite directions.

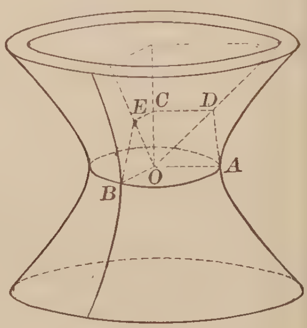
(iii) Plane sections parallel to the plane xy are hyperbolas whose transverse axes are parallel to the axis of x, or y, according as z is positive or negative. The trace on this plane is two intersecting right lines.

Diagrams.

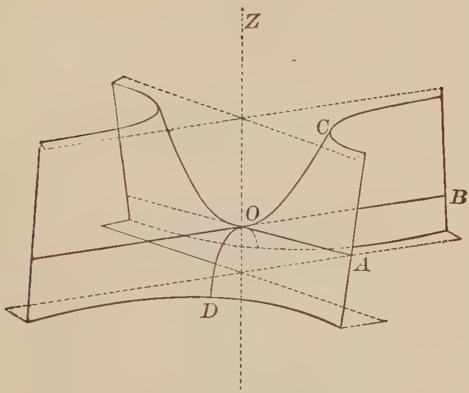
Note. These figures are taken, by permission, from W. B. Smith's Geometry.



Elliptic Paraboloid.
$FA = 2a$ and $GB = 2b$ are half-parameters.



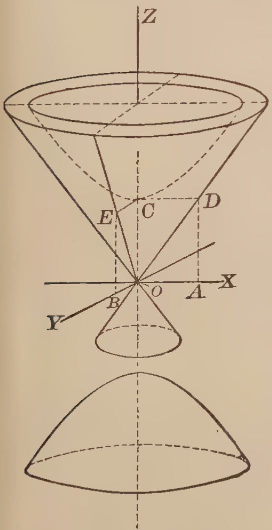
Simple Hyperboloid.
AB is Ellipse of the Gorge.
EOD is the Asymptotic Cone.



HYPERBOLIC PARABOLOID.

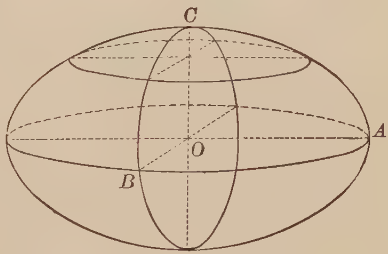
OC and OD are Parabolas.

OA and OB are Asymptotic directions for the Hyperbolas.



DOUBLE HYPERBOLOID.

EOD is the Asymptotic Cone.



ELLIPSOID.

$OA = a,\ OB = b,\ OC = c.$

ANSWERS.

Exercise 3. Page 7.

1. Let $x_1 = -2$, $y_1 = 5$, $x_2 = -8$, $y_2 = -3$. Substituting in [1], we have

$$d = \sqrt{(-6)^2 + (-8)^2} = \sqrt{100} = 10.$$

In Fig. 3, the points P and Q are plotted to represent this case. If we choose to solve the question without the aid of [1], we may neglect algebraic signs, and we have

$$QR = NO - MO = 8 - 2 = 6;$$
$$PR = PM + MR = 5 + 3 = 8;$$
$$\therefore \overline{PQ}^2 = \overline{QR}^2 + \overline{PR}^2 = 36 + 64 = 100, \text{ and } PQ = 10.$$

2. 13.

3. 5.

4. 10.

5. $2\sqrt{a^2 + b^2}$.

6. 25, 29, $20\sqrt{2}$.

7. $2\sqrt{17}$, $5\sqrt{2}$, $\sqrt{106}$.

8. 5, 5, 6.

9. a, b, $\sqrt{a^2 + b^2}$.

10. $\sqrt{29}$, 5, $2\sqrt{10}$, $4\sqrt{5}$; $2\sqrt{10}$, $3\sqrt{13}$.

11. 8 or -16.

12. $(x - 7)^2 + (y + 2)^2 = 121$.

13. $(x - 2)^2 + (y - 3)^2 = (x - 4)^2 + (y - 5)^2$, which reduces to $x + y = 7$.

Exercise 4. Page 9.

1. (6, 6).

2. $(-1, 0)$.

3. $(2, -2)$.

4. $(3, -1)$, $(\frac{1}{2}, -\frac{11}{2})$, $(-\frac{1}{2}, -\frac{3}{2})$.

5. (7, 1).

6. $(a, -b)$.

7. Take the origin of coördinates at the intersection of the two legs, and the axes of x and y in the directions of the legs. Then, if a and b denote the lengths of the legs, the coördinates of the three vertices will be (0, 0), $(a, 0)$, and $(0, b)$.

10. Observe that now the distances RB and BQ will be $x - x_2$ and $y - y_2$.

11. (6, 2).

12. $(\frac{8}{3}, \frac{1}{3})$.

13. (8, 0).

14. $(7\frac{3}{4}, -31\frac{3}{4})$.

15. $(13, -1)$, $(-11, 5)$, $(1, -11)$.

Exercise 7. Page 23.

1. 12, 16.
2. -10, 6.
3. ± 4, ± 4.
4. $\pm \frac{4}{3}$, ± 2.
5. $\pm \frac{4}{3}$, imaginary.
6. $\pm \frac{4}{3}$, -4.
7. $\pm b$, $\pm a$.
8. 3 on axis of x.
9. ± 3 on axis of x.
10. Locus passes through origin.
11. Locus passes through origin.
12. $\begin{cases} \text{On axis of } x, 8, \text{ and } -4. \\ \text{On axis of } y, 4 \pm 4\sqrt{3}. \end{cases}$
13. $\begin{cases} \text{On axis of } x, 0 \text{ and } 4. \\ \text{On axis of } y, 0 \text{ and } 8. \end{cases}$
14. Locus does not cut the axes.
15. (5, 7).
16. (2, 1).
17. (3, 4) and $(-4, 3)$.
18. (3, 4).
19. (5, 3) and (3, 5).
20. (0, 0) and (2, 4).
21. $(5, -3)$, $(6, 4)$, $(-4, -1)$.
22. $\sqrt{61}$, 5, $2\sqrt{26}$.
23. 3, 4, 5.
24. $\begin{cases} (a, b)\ (-a, b), \\ (-a, -b)\ (a, -b). \end{cases}$
25. No.
26. 10.

Exercise 9. Page 31.

1. Let x and y denote the variable coördinates of the moving point. Then it is evident that for all positions of the point $y = 3x$. Therefore, the required equation is $y = 3x$, or $y - 3x = 0$. Does the locus of this equation pass through the origin?

2. $x - 6 = 0$, $x + 6 = 0$, $x = 0$.

3. $y - 4 = 0$, $y + 1 = 0$, $y = 0$.

4. The line $x = 3$ is the line AB (Fig. 74); how is this line drawn? The locus of the variable point consists of the two parallels to AB, drawn at the distance 2 from AB. Let CD, EF, be these parallels, and (x, y) denote in general the variable point, then, for all points in CD, $x = 3 + 2 = 5$, and for all points in EF, $x = 3 - 2 = 1$. Therefore, the equation of the line CD is $x - 5 = 0$, and that of the line EF is $x - 1 = 0$. The product of these two equations is the equation $(x - 5)(x - 1) = 0$. This equation is evidently satisfied by every point in each of the lines CD and EF, and by no other points. Therefore, the required equation is $(x - 5)(x - 1) = 0$, or $x^2 - 6x + 5 = 0$. Verify that this equation is satisfied by points taken at random in the lines CD and EF.

5. $y^2 - 10y + 16 = 0$, two parallel lines.

6. $x^2 + 8x - 9 = 0$, two parallel lines. **7.** $x + 3 = 0$, $y - 2 = 0$.

8. It is proved in elementary geometry that all points equidistant from two given points lie in the perpendicular erected at the middle point of the line joining the two given points. This perpendicular is the locus required, and its equation evidently is $x = 3$.

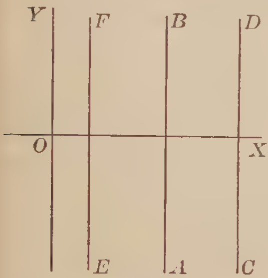


Fig. 74.

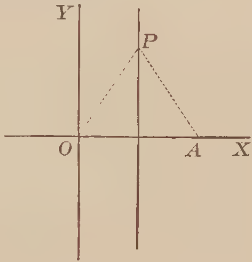


Fig. 75.

Let us now solve this problem by the analytic method. Let O (Fig. 75) be the origin, A the point $(6, 0)$, and let P represent any position of a point equidistant from O and A, x and y its two coördinates.

Then from the given condition

$$PO = PA.$$

Therefore,
$$x^2 + y^2 = (x - 6)^2 + (y - 0)^2,$$
or
$$x^2 + y^2 = x^2 - 12x + 36 + y^2;$$
whence
$$x = 3,$$
the equation of the locus required.

9. $x - 1 = 0$. **10.** $y - 2 = 0$. **11.** $x - 3y - 1 = 0$. **12.** $x - y = 0$.

13. $x^2 + y^2 = 100$, a circle with the origin for centre and 10 for radius.

14. Express by an equation the fact that the distance from the point (x, y) to the point $(4, -3)$ is equal to 5. The equation is $(x - 4)^2 + (y + 3)^2 = 25$.

15. $(x + 4)^2 + (y + 7)^2 = 64$. **16.** $x^2 + y^2 = 81$.

17. Draw $AO \perp$ to BC (Fig. 76). Take AO for the axis of x, and BC for the axis of y; then A is the point $(3, 0)$.

Let P represent any position of the vessel, x and y its coördinates

OM and PM. Join PA, and draw $PQ \perp BC$, and meeting it in Q. Then from the given condition

$$PA = PQ = OM.$$

Therefore, $$\overline{PA}^2 = \overline{OM}^2.$$

Now $\overline{PA}^2 = \overline{AM}^2 + \overline{PM}^2 = (x-3)^2 + y^2$, ad $\overline{OM}^2 = x^2$. Substituting, we have $(x-3)^2 + y^2 = x^2$;

whence $$y^2 = 6x - 9.$$

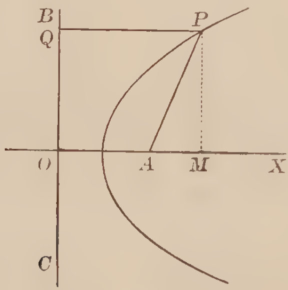


Fig. 76.

The locus is thé curve called the parabola. We leave the discussion of the equation as an exercise for the learner.

18. If BC is taken for the axis of y, and the perpendicular from A to BC as the axis of x, the required equation is $y^2 = 12x - 36$.

19. $x^2 - 3y^2 = 0$, two straight lines.

20. $x^2 + y^2 = k^2 - a^2$, a circle.

21. $4ax \pm k^2 = 0$, two straight lines.

Exercise 10. Page 33.

4. $d = \sqrt{x_1^2 + y_1^2}$.

5. $(x-4)^2 + (y-6)^2 = 64$.

6. $x + y = 7$.

7. $(\frac{13}{3}, \frac{8}{3})$; $\frac{5}{3}\sqrt{2}$.

8. Take two sides of the rectangle for the axes, and let a and b represent their lengths; then the vertices of the rectangle will be the points $(0, 0)$, $(a, 0)$, (a, b), $(0, b)$.

9. Take one vertex as the origin, and one side, a, as the axis of x; then $(0, 0)$ and $(a, 0)$ will be two vertices. Let (b, c) be a third vertex; then $(a + b, c)$ will represent the fourth.

10. $(11, 2), (-1, 4), (15, 16)$.

11. $(5, -2), (\frac{5}{2}, \frac{1}{2}), (\frac{9}{2}, -\frac{13}{2})$.

12. $(1, -\frac{8}{3})$.

13. $\sqrt{17}$.

14. $(\frac{7}{2}, \frac{7}{2})$.

16. $(6, 23)$.

17. $\left(\dfrac{x_1 + 3x_2}{4}, \dfrac{y_1 + 3y_2}{4}\right), \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right), \left(\dfrac{3x_1 + x_2}{4}, \dfrac{3y_1 + y_2}{4}\right)$.

21. 3 or -23.

22. $\begin{cases} 3 \text{ and } 2 \text{ on } OX. \\ 6 \text{ and } 1 \text{ on } OY. \end{cases}$

23. $(8, 6)$ and $(8, -6)$.

24. $(2a, a)$ and $(-2a, a)$.

25. $(a, 0)$ and $(-a, 0)$.

26. $10, 2\sqrt{26}, 2\sqrt{13}$.

27. Taking the fixed lines for axes, the equation is $y = 6x$, or $x = 6y$.

28. Taking A for origin, and AB for the axis of x, the equation is $x^2 - 3y^2 = 0$.

29. Taking the fixed line and the perpendicular to it from the fixed point as the axes of x and y respectively, the required equation is $x^2 + (y - a)^2 = 4y^2$.

Exercise 11. Page 40.

1. $x - y + 1 = 0$.

2. $2x - y - 3 = 0$.

3. $x + y - 1 = 0$.

4. $x - y = 0$.

5. $3x + 2y - 12 = 0$.

6. $2x - 3y + 6 = 0$.

7. $x + y - 7 = 0$.

8. $4x - 3y = 0$.

9. $y = 0$.

10. $y = 4$.

11. $5x - 2y = 0$.

12. $nx - my = 0$.

13. $x - y - 3 = 0$.

14. $\sqrt{3}x - y + 7 - 2\sqrt{3} = 0$.

15. $x - y + 14 = 0$.

16. $\sqrt{3}x + 3y + 12 - 13\sqrt{3} = 0$.

17. $\sqrt{3}x - 3y - 3\sqrt{3} = 0$.

18. $x + y - 3 = 0$.

19. $\sqrt{3}x + y = 0$.

20. $y + 3 = 0$.

21. $x - 2 = 0$.

22. $x - y + 2 = 0$.

23. $x - y + 5 = 0$.

24. $x - y - 4 = 0$.

25. $x - \sqrt{3}y - 4\sqrt{3} = 0$.

26. $y + 4 = 0$.

27. $\sqrt{3}x - y - 4 = 0$.

28. $x = 0$.

29. $\sqrt{3}x + y + 4 = 0$.

30. $x + y + 4 = 0$.

31. $x + \sqrt{3}y + 4\sqrt{3} = 0$.

32. $y + 4 = 0$.

33. $3x + 4y - 12 = 0$.

34. $x - 3y + 6 = 0$.

35. $x + y + 3 = 0$.

36. $3x - 5y - 15 = 0$.

37. $x - 2y + 10 = 0$.

38. $x - y - 1 = 0$.

39. $x - y - n = 0$.

40. $4x + y - 4n = 0$.

41. $x + y - 5\sqrt{2} = 0$.

42. $x - y\sqrt{3} + 10 = 0$.

43. $x + y\sqrt{3} + 10 = 0$.

44. $x - y\sqrt{3} - 10 = 0$.

45. $\begin{cases} x + 7y + 11 = 0,\ x - 3y \\ + 1 = 0,\ 3x + y - 7 = 0. \end{cases}$

46. $\begin{cases} x - 7y = 39,\ 9x - 5y = 3, \\ 4x + y = 11. \end{cases}$

47. $\begin{cases} 17x - 3y = 25,\ 7x + 9y \\ = -17,\ 5x - 6y - 21 = 0. \end{cases}$

48. $\begin{cases} 5x - y = 0,\ 5x + 6y - 35 = 0, \\ 3x - y = 21,\ 9x + 4y = 0, \\ y = 0,\ 14x + 3y = 29. \end{cases}$

49. $x - y\sqrt{3} - 7\frac{1}{3} = 0$.

50. $y = x + 3$.

51. $y = x \pm 6\sqrt{2}$.

52. $y = -x \pm 6\sqrt{2}$.

53. $\dfrac{x}{-\frac{11}{3}} + \dfrac{y}{\frac{11}{2}} = 1$.

55. $a = \dfrac{C}{A}$, or $-\dfrac{b}{m}$, $b = \dfrac{C}{B}$.

58. $m = -\dfrac{A}{B}$, or $-\dfrac{b}{a}$, $b = \dfrac{C}{B}$.

59. $(5, -3)$, $(6, 4)$, $(-4, -1)$.

60. $9x + 2y = 0$, $\frac{1}{6}\sqrt{85}$.

61. $y \pm x = y_1 \mp x_1$.

62. $\begin{cases} (d-c)x-(b-a)y=ad-bc, \\ (d-c)x+(b-a)y=bd-ac. \end{cases}$

64. $\begin{cases} 2y_2x+(x_1-2x_2)y-x_1y_2=0, \\ y_2x+(2x_1-x_2)y-x_1y_2=0, \\ y_2x-(x_1+x_2)y=0. \end{cases}$

65. $m = 4$.

66. $m = 3$.

67. $b = -9$.

68. $\dfrac{y_3 - y_1}{x_3 - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$, or $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$.

Exercise 12. Page 44.

1. $-\frac{3}{13}\sqrt{13}x + \frac{2}{13}\sqrt{13}y = \frac{11}{13}\sqrt{13}$; $p = \frac{11}{13}\sqrt{13}$.

2. $\frac{3}{34}\sqrt{34}x + \frac{5}{34}\sqrt{34}y = \frac{13}{34}\sqrt{34}$; $p = \frac{13}{34}\sqrt{34}$.

3. $p = \frac{2}{17}\sqrt{17}$.

4. $p = \frac{7}{13}\sqrt{13}$.

5. $p = \frac{1}{2}\sqrt{26}$.

6. $p = \frac{19}{10}\sqrt{2}$.

7. $p = \dfrac{n}{\sqrt{e^2 + c^2}}$.

8. $p = \dfrac{r}{\sqrt{n^2 + c^2}}$.

9. Fourth quadrant.

10. Second quadrant.

11. Fourth quadrant.

12. Second quadrant.

13. Third quadrant.

14. First quadrant.

15. Second quadrant.

16. Fourth quadrant.

17. Third quadrant.

18. Fourth quadrant.

20. $m = -\frac{b}{a}$.

21. 0; 8.

22. $C = 12$, $A = 4$, $B = -1$.

24. $A = (y_2 - y_1)$, $B = -(x_2 - x_1)$, $C = (x_1y_2 - x_2y_1)$

25. $m = \frac{y_2 - y_1}{x_2 - x_1}$, $b = \frac{x_2y_1 - x_1y_2}{x_2 - x_1}$.

Exercise 13. Page 46.

1. $3x - y - 16 = 0$.
2. $3x - 4y - 3 = 0$.
3. $4x - y = 0$.
4. $y - 8 = 0$.
5. $x - 5 = 0$.
6. $x + 4y + 49 = 0$.
7. $7x - 23y + 193 = 0$.
8. $y = 2x$.
9. $35y + 49x - 79 = 0$.

Exercise 14. Page 47.

2. $\tan\phi = -\frac{1}{2}$.

3. $\tan\phi = \frac{1}{18}$.

4. $\tan\phi = \frac{n}{n^2 + 2}$.

5. 90°. 6. 135°. 7. 90°. 8. 0°. 9. 30°.

11. $\begin{cases} y = 5x - 10, \\ x + 5y = 28. \end{cases}$

12. $\begin{cases} y = 5x + 11, \\ x + 5y - 3 = 0. \end{cases}$

13. $\begin{cases} y - 3 = m'(x - 2), \\ \text{and } m' = -(8 \pm 5\sqrt{3}). \end{cases}$

14. $\begin{cases} y - 3 = m'(x - 1), \\ \text{and } m' = \frac{8 \pm 5\sqrt{3}}{11}. \end{cases}$

22. $2x + 3y - 31 = 0$.

23. $62x + 31y - 1115 = 0$.

24. $y = 6x - 27$.

25. $y = mx \pm d\sqrt{1 + m^2}$.

26. $Bx = A(y - b)$.

27. $ax - by = a^2 - b^2$.

28. $(a \pm b)y + (b \mp a)(x - a) = 0$.

30. $\begin{cases} x - 3y + 26 = 0, \\ 5x + 3y + 8 = 0, \\ 2x + 3y - 9 = 0. \end{cases}$

31. $x - 6 = 0$.

32. $\begin{cases} 2x - 9y + 12 = 0, \\ 10x - 4y + 63 = 0, \\ 18x - 40y + 111 = 0. \end{cases}$

33. $\left.\begin{cases} x - y - 6 = 0 \\ 2x - y - 2 = 0 \\ 5x - 3y - 10 = 0 \end{cases}\right\}$ meeting in the point. $(-4, -10)$. Distance $= \sqrt{85}$.

35. $y - y_1 = \frac{-A \pm B\tan\phi}{B \pm A\tan\phi}(x - x_1)$.

Exercise 15. Page 52.

1. $\frac{1}{2}\sqrt{10}$. 2. $\frac{4}{5}\sqrt{5}$. 3. 4. 4. $\frac{3}{5}\sqrt{5}$. 5. 0.

7. $-\frac{24}{5}, -\frac{20}{5}, -\frac{16}{5}, -\frac{12}{5}, -\frac{8}{5}, -\frac{4}{5}, 0, +\frac{4}{5}, -\frac{15}{5}, -\frac{12}{5}, -\frac{9}{5}, -\frac{6}{5}, -\frac{3}{5}, 0, +\frac{3}{5}$. The learner should construct the given lines, and observe how the *sign* of the required distance gives the direction of the point from the line.

8. $-6, -5, -4, 3, 2, 1, 0, -1$. The learner should construct the lines, and observe the change of sign of the distance, as in No. 7.

9. $-\frac{9}{5}\sqrt{10}$.

10. $\frac{20}{41}\sqrt{41}$.

11. $2\sqrt{2}$.

12. $-\frac{3}{8}\sqrt{2}$.

13. $-\frac{11}{5}$.

14. $\frac{6}{13}\sqrt{13}$.

15. $-\frac{31}{5}\sqrt{2}$.

16. $\mp\frac{4}{5}\sqrt{2}$.

17. $\sqrt{a^2+b^2}$.

18. $\dfrac{\pm ab}{\sqrt{a^2+b^2}}$; $\dfrac{\mp 3ab}{\sqrt{a^2+b^2}}$.

19. $\sqrt{a^2+b^2}$.

20. $\pm\dfrac{Ah+Bk-(D-C)}{\sqrt{A^2+B^2}}$.

21. 2.

22. 4.

23. $\pm\dfrac{C-C'}{\sqrt{A^2+B^2}}$.

24. $\dfrac{C+C'}{\sqrt{A^2+B^2}}$.

25. $\frac{5}{13}\sqrt{26}$.

26. $\pm\dfrac{3ab}{2\sqrt{a^2+b^2}}$.

Exercise 16. Page 54.

1. $1\frac{1}{2}$.

2. 12.

3. 29.

4. 40.

5. ab.

7. 26.

8. 35.

9. $19\frac{1}{2}$.

10. $\frac{1}{2}(x_1y_2-x_2y_1)$.

11. 26.

12. 96.

13. 41.

14. $\frac{1}{2}(a-c)(b-1)$.

15. $\frac{1}{2}(a-b)(a+b-2c)$.

16. $\frac{1}{2}(a^2-b^2)$.

17. $60°, 60°, 60°$; $9\sqrt{3}$.

18. 10.

19. $\frac{1}{2}$.

20. $1\frac{1}{2}$.

21. $9a^2$.

22. $\dfrac{2c^2}{21}$.

23. 24.

24. 36.

25. 16.

26. $\frac{1}{2}ab$.

27. $\dfrac{b^2}{2m}$.

28. $\frac{1}{2}ab$.

29. $\dfrac{C^2}{2AB}$.

30. 56.

31. $10\frac{1}{2}$.

Exercise 17. Page 56.

3. $2,\ \infty,\ 90^\circ,\ 2,\ 0^\circ.$

4. $0,\ 0,\ 45^\circ,\ 0,\ 135^\circ.$

5. $\frac{1}{3}\sqrt{3}-2,\ 2\sqrt{3}-1,\ 60^\circ,\ \frac{2\sqrt{3}-1}{2},\ 150^\circ.$

6. $2,\ \frac{2}{3}\sqrt{3},\ 150^\circ,\ 1,\ 60^\circ.$

7. $2,\ -\frac{2}{3}\sqrt{3},\ 30^\circ,\ 1,\ 300^\circ.$

8. $\frac{2}{3}\sqrt{3},\ -2,\ 60^\circ,\ 1,\ 330^\circ.$

9. $\begin{cases} 11x + y = 0, \\ x - 5y + 20 = 0. \end{cases}$

10. $\frac{4}{41}\sqrt{82}.$

11. $\begin{cases} 3x + 4y - 57 = 0, \\ 3x + 4y + 6 = 0, \\ 12x - 5y - 39 = 0, \\ 12x - 5y + 24 = 0. \\ \text{Area} = 63. \end{cases}$

12. 43.

13. $x = 3.$

14. $\begin{cases} x - y + 1 = 0, \\ x + y - 7 = 0. \end{cases}$

15. $5x + 6y - 39 = 0.$

16. $14x - 3y - 30 = 0.$

17. $4x - 5y + 8 = 0.$

18. $x + y - 7 = 0.$

19. $\frac{y - y_3}{x - x_3} = \frac{y_2 - y_1}{x_2 - x_1}.$

20. $\begin{cases} y = 3,\ 13y = 5x - 1, \\ 9y = 5x + 7. \end{cases}$

21. $92x + 69y + 102 = 0.$

22. $x + 4y = 34.$

23. $3x + 4y - 5a = 0.$

24. $3x + 4y = 24.$

25. $y - y_1 = -\frac{y_1}{x_1}(x - x_1).$

26. $4y = x + 8.$

27. $4y = 9x - 24.$

28. $\begin{cases} 9x - 20y + 96 = 0, \\ 5x - 4y + 32 = 0. \end{cases}$

29. $88x - 121y + 371 = 0.$

30. $\begin{cases} 5x - y - 10 = 0, \\ x + 5y - 28 = 0. \end{cases}$

31. $\begin{cases} 2x + y - 9 = 0, \\ x - 2y - 17 = 0. \end{cases}$

32. $\begin{cases} 4x + y - 20 = 0, \\ x - 4y - 5 = 0. \end{cases}$

33. $2x = y,\ 2y = x.$

34. $4x + 5y + 11 \pm 3\sqrt{41} = 0.$

35. $y = (7 \mp 5\sqrt{2})(x + 2).$

36. $\frac{2x - 5y}{\sqrt{29}} = \pm\frac{4x + 3y - 12}{5}.$

37. $\begin{cases} 7x - 3y + 15 = 0, \\ 3x + 7y - 93 = 0. \end{cases}$

38. $\begin{cases} 8x + 7y - 19 = 0, \\ 16x + 3y + 17 = 0. \end{cases}$

39. $135^\circ.$

40. $90^\circ.$

41. $-\frac{31\sqrt{26}}{143}.$

42. $\pm\frac{bh + ak - ab}{\sqrt{a^2 + b^2}}.$

43. $\frac{c^2}{\sqrt{h^2 + k^2}}.$

44. $\pm\frac{a}{m}\sqrt{1 + m^2}.$

45. $c^2.$

46. $\frac{k^2}{6}$.

47. $\frac{2a^2+5ab+2b^2}{6}$.

48. $17\frac{1}{2}$.

49. $6\frac{17}{48}$.

50. 59.

51. $(10, 5\frac{1}{3})$.

54. xy represents the two axes.

57. $a=5$.

58. $x+a=0$, $x-b=0$.

59. $x+a=0$, $y+b=0$.

60. The axes and $x=y$.

61 $2x-y=0$, $7x+y=0$.

62. If h denotes the altitude of the triangle, and the base is taken as the axis of x, the locus is the straight line $y=h$.

63. The equation of the locus is

$$(x-x_1)^2+(y-y_1)^2=(x-x_2)^2+(y-y_2)^2.$$

This is the equation of the straight line bisecting the line joining (x_1, y_1) and (x_2, y_2), and $\perp$ to it.

64. The two parallel lines represented by

$$Ax+By+C\pm d\sqrt{A^2+B^2}=0.$$

65. $x+y=k$.

66. $\frac{Ax+By+C}{\sqrt{A^2+B^2}}+\frac{A'x+B'y+C'}{\sqrt{A'^2+B'^2}}=k$.

67. Let b denote the base, k^2 the constant difference of the squares of the other two sides. Taking the base as the axis of x, and the middle point of the base as origin, the equation of the locus is $2bx=\pm k^2$.

Exercise 18. Page 64.

1. $7x+y=0$.

2. $x+2y-13=0$.

3. $5x+6y-37=0$.

4. $\begin{cases} x-y+8=0, \\ x+y-6=0. \end{cases}$

5. $y=x+3$.

6. $64x-23y=59$.

7. $44x+y=0$.

8. $5x+y-16=0$.

9. $(AC'-A'C)x+(BC'-B'C)y=0$.

10. $(BA'-AB')y+CA'-AC'=0$.

11. $\frac{Ax+By+C}{Ax_1+By_1+C}=\frac{A'x+B'y+C'}{A'x_1+B'y_1+C'}$.

12. $472x-29y+174=0$.

13. $y=x\sqrt{3}+3-\sqrt{3}$.

14. $\begin{cases} 4x+3y-25=0, \\ 3x-4y+25=0. \end{cases}$

15. $\frac{y}{a}-\frac{x}{b}=\frac{mb-a}{ma+b}$.

16–18. Generally the easiest way to solve such exercises as these is to find the intersection of two of the lines, and then substitute its coördinates in the equation of the third line.

19. $m=1$.

20. When $\dfrac{m''-m}{m''-m'}=\dfrac{b''-b}{b''-b'}$.

21. If we choose as axes one side of the triangle and the corresponding altitude, we may represent the three vertices by $(a, 0)$, $(-c, 0)$, $(0, b)$.

22. Choosing as axes one side and the perpendicular erected at its middle point, the vertices may be represented by $(a, 0)$, $(-a, 0)$, (b, c).

23. It is well here to choose the same axes as in No. 21.

24. Choosing the origin anywhere within the triangle, it is evident that the equations of the bisectors in the normal form may be written as follows:

$$\begin{aligned}
(x\cos\alpha + y\sin\alpha - p) - (x\cos\alpha' + y\sin\alpha' - p') &= 0,\\
(x\cos\alpha' + y\sin\alpha' - p') - (x\cos\alpha'' + y\sin\alpha'' - p'') &= 0,\\
(x\cos\alpha'' + y\sin\alpha'' - p'') - (x\cos\alpha + y\sin\alpha - p) &= 0.
\end{aligned}$$

Now, by adding any two of these equations, we obtain the third; therefore, the three bisectors must pass through one point.

25. $\begin{cases} 2\sqrt{2},\ \sqrt{10},\ 2\sqrt{10}. \\ \text{Origin within the } \triangle. \end{cases}$

26. $\frac{18}{5}\sqrt{10}$, $\frac{18}{17}\sqrt{34}$, $\frac{18}{13}\sqrt{13}$.

27. $\begin{cases} x+y+10=0, \\ 7x-7y+24=0. \end{cases}$

28. $\begin{cases} 7x-9y+34=0, \\ 9x+7y-12=0. \end{cases}$

29. $\begin{cases} x-y+2=0, \\ x+y-14=0. \end{cases}$

30. $\begin{cases} x-1=0, \\ y-1=0. \end{cases}$

31. $\dfrac{\pm(y-mx-b)}{\sqrt{1+m^2}}=\pm\dfrac{\pm(y-m'x-b')}{\sqrt{1+m'^2}}$.

Exercise 19. Page 68.

1. (i) Parallel to the axis of x, (ii) parallel to the axis of y.

2. When $ad=bc$.

3. The two lines are *real*, *imaginary*, or *coincident*, according as C^2-4AB is *positive*, *negative*, or *zero*. The two lines are $\perp$ to each other when $A+B=0$.

5. $x+y+1=0$, and $x-3y+1=0$.

6. $x-2y\pm(y-3)\sqrt{-1}=0$.

7. $x-y-3=0$, and $x-3y+3=0$.

8. $45°$.

9. $K=2$.

10. $K=-10$, or $-\frac{35}{2}$.

11. $K=28$.

12. $K=\frac{15}{2}$.

Exercise 20. Page 70.

1. Take the point O as origin, and the axis of y parallel to the given lines. If the equations of the given lines are $x = a$, $x = b$, and if the slopes of the lines drawn in the two fixed directions are denoted by m', m'', the equation of the locus is

$$(b-a)\,y = m'b\,(x-a) - m''a\,(x-b).$$

2. If a and b are the sides of the right triangle, the equation of the locus is

$$y = \pm\frac{a}{b}x.$$

3. Let $OA = a$, $OB = b$. Then the equation of the locus is

$$x + y = a + b.$$

4. Take as axes the base and the altitude of the triangle. Let a and b denote the segments of the base, h the altitude. Then the equation of the locus is

$$\frac{2x}{b-a} + \frac{2y}{h} = 1.$$

This is a straight line joining the middle points of the base and the altitude.

5. Take as axes the sides of the rectangle, and let a, b denote their lengths. The equation of the locus is

$$bx - ay = 0.$$

Hence, the locus is a diagonal of the rectangle.

Exercise 21. Page 73.

1. $x^2 + y^2 = -2rx$.

2. $x^2 + y^2 = 2ry$.

3. $x^2 + y^2 = -2ry$.

4. $(x-5)^2 + (y+3)^2 = 100$.

5. $x^2 + (y+2)^2 = 121$.

6. $(x-5)^2 + y^2 = 25$.

7. $(x+5)^2 + y^2 = 25$.

8. $(x-2)^2 + (y-3)^2 = 25$.

9. $x^2 + y^2 - 2hx - 2ky = 0$.

11. $(1, 2)$, $\sqrt{5}$.

12. $(\frac{5}{6}, \frac{7}{6})$, $\frac{1}{6}\sqrt{62}$.

13. $(4, 0)$, 4.

14. $(-4, 0)$, 4.

15. $(0, 4)$, 4.

16. $(0, -4)$, 4.

17. $(0, \frac{7}{6})$, $\frac{7}{6}$.

18. $(0, 0)$, $3k$.

19. $(0, 0)$, $2k$.

20. $(0, 0)$, $\sqrt{a^2 + b^2}$.

21. $\left(\frac{k}{2}, 0\right)$, $\frac{k}{2}\sqrt{5}$.

22. $\left(\frac{h}{2}, \frac{k}{2}\right)$, $\frac{\sqrt{h^2 + k^2}}{2}$.

23. When $D = D'$ and $E = E'$; in other words, when the two equations differ only in their constant terms.

24. In this case, $r = 0$. Hence, the equation represents simply the point (a, b). We may also say that it is the equation of an infinitely small circle, having this point for centre.

26. $\begin{cases} (\frac{5}{2}, \frac{7}{2}),\ \frac{5}{2}\sqrt{2}; \\ \text{On } OX,\ 3 \text{ and } 2; \\ \text{On } OY,\ 6 \text{ and } 1. \end{cases}$

27. $\begin{cases} (6, 2),\ 5; \\ \text{On } OX,\ 6 \pm \sqrt{21}; \\ \text{On } OY, \text{ imaginary points.} \end{cases}$

28. $\begin{cases} (2, 4),\ 2\sqrt{5}; \\ \text{On } OX,\ 0 \text{ and } 4; \\ \text{On } OY,\ 0 \text{ and } 8. \end{cases}$

29. $\begin{cases} (3, -2),\ 3; \\ \text{On } OX,\ 3 \pm \sqrt{5}; \\ \text{On } OY,\ -2. \end{cases}$

30. $\begin{cases} (-11, 9),\ \sqrt{145}; \\ \text{On } OX,\ -3 \text{ and } -19; \\ \text{On } OY,\ 9 \pm 2\sqrt{6}. \end{cases}$

31. $\begin{cases} \text{(i) } D^2 = 4C. \\ \text{(ii) } E^2 = 4C. \\ \text{(iii) } 4C > D^2 \text{ and } E^2. \end{cases}$

32. $x^2 + y^2 + 10x + 10y + 25 = 0$.

33. $(7, 4)$ and $(8, 1)$.

34. $(2, 0)$ and $(\frac{6}{5}, -\frac{8}{5})$.

35. $\frac{4}{5}\sqrt{5}$.

36. $2\left(r^2 - \dfrac{a^2b^2}{a^2 + b^2}\right)^{\frac{1}{2}}$.

37. $2x - y - 2 = 0$.

38. $4x - 5y - 71 = 0$.

39. $3x - 5y - 34 = 0$.

40. Let (x, y) be any point in the required locus; then the distance of (x, y) from (x_1, y_1) must always be equal to its distance from (x_2, y_2);

therefore, $\quad (x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2$;

whence $\quad 2x(x_1 - x_2) + 2y(y_1 - y_2) = (x_1^2 + y_1^2 - x_2^2 - y_2^2)$.

Show that this represents a straight line $\perp$ to the line joining (x_1, y_1) and (x_2, y_2) at its middle point.

41. $8x + 6y + 17 = 0$.

42. First Method. Substitute successively the coördinates of the given points in the general equation of the circle; this gives three equations of condition, and by solving them we find the values of a, b, r.

Second Method. Join $(4, 0)$ to $(0, 4)$ and also to $(6, 4)$ by straight lines, then erect perpendiculars at the middle points of these two lines; their intersection will be the centre of the circle, and the distance from the centre to either one of the given points will be the radius.

Ans. $x^2 + y^2 - 6x - 6y + 8 = 0$.

43. $x^2 + y^2 - 8x + 6y = 0$.

44. $x^2 + y^2 + 6x + y = 0$.

45. $x^2 + y^2 + 8ax - 6ay = 0$.

46. $x^2 + y^2 + 8x + 20y + 31 = 0$.

47. $x^2 + y^2 - 9x - 5y + 14 = 0.$

48. $\begin{cases} (x - 5)^2 + (y + 8)^2 = 169, \\ (x - 22)^2 + (y - 9)^2 = 169. \end{cases}$

49. $\begin{cases} x^2 + y^2 - 30(x + y) + 225 = 0, \\ x^2 + y^2 - 6(x + y) + 9 = 0. \end{cases}$

50. $x^2 + y^2 - 8x - 8y + 16 = 0.$

51. $x^2 + y^2 \mp 2ax \mp 2ay + a^2 = 0.$

52. $x^2 + y^2 = ax + by.$

53. $(x - 1)^2 + (y - 4)^2 = 20.$

54. $x^2 + y^2 - 14x - 4y - 5 = 0.$

55. $x^2 + y^2 \pm \sqrt{2}ay = 0,$
$x^2 + y^2 \pm \sqrt{2}ax = 0.$

56. $m(x^2 + y^2) - ab = (ma - b)x + (mb - a)y.$

57. $x^2 + y^2 = x_1 x + y_1 y.$

58. $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$

59. $(1 + m^2)(x^2 + y^2) - 2r(x + my) = 0.$

60. $x^2 - ax + y^2 = r^2 - \frac{a^2}{2}.$

Exercise 22. Page 81.

1. The double sign corresponds to the geometric fact that *two* tangents having the same direction may always be drawn to a given circle.

3. $2x + 3y = 26,\ 3x - 2y = 0;\ 3\sqrt{13},\ 2\sqrt{13},\ -9,\ -4,\ \frac{13}{3}\sqrt{13}.$

4. $\frac{x_1^2 - r^2}{x_1},\quad -x_1,\quad \frac{r^3}{x_1 y_1}.$

5. $9x - 13y = 250.$

6. $x \pm 3y = 10.$

7. $104\frac{1}{6}.$

8. $x^2 + y^2 = 25\frac{1}{2}.$

9. $14x \pm 6y = 232.$

10. $3x + y = 19.$

11. $3x + 4y = 0.$

12. $\begin{cases} 3x + 7y = 93, \\ 3x - 7y = 65. \end{cases}$

13. $x = r.$

14. $Ax + By \mp r\sqrt{A^2 + B^2} = 0.$

15. $Bx - Ay \mp r\sqrt{A^2 + B^2} = 0.$

16. $x - y \pm r\sqrt{2} = 0.$

17. The equation of the two tangents is $(h^2 - r^2)y^2 = r^2(x - h)^2.$

18. $x + y = \pm r\sqrt{2}.$

19. $\begin{cases} x = 10, \\ 3x + 4y = 50. \end{cases}$

20. $y = 2x + 13 \pm 6\sqrt{5}.$

21. $-21,\ -3\frac{6}{7}.$

22. $\begin{cases} x^2 + y^2 = p^2, \\ (p \cos \alpha,\ p \sin \alpha). \end{cases}$

23. $\begin{cases} \text{When } C = r\sqrt{A^2 + B^2}. \\ \text{When } \dfrac{Aa + Bb - C}{\sqrt{A^2 + B^2}} = \pm r. \end{cases}$

24. $ax + by = 0.$

25. $(-a,\ -b).$

26. $(2a,\ b).$

27. $(0,\ b).$

28. $x^2 + y^2 = \frac{5}{2}.$

29. $m = 0.$

30. $c = -36 \mp 20\sqrt{6}.$

31. $(x - 5)^2 + (y - 3)^2 = \frac{121}{13}.$

32. $\begin{cases} (x - 2)^2 + (y - 4)^2 = 100, \\ (x - 18)^2 + (y - 16)^2 = 100. \end{cases}$

33. $(x - 1)^2 + (y - 6)^2 = 25.$

34. $\frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2}.$

35. $(x^2 + y^2)(a + b + \sqrt{a^2 + b^2})^2.$
$- 2ab(a + b + \sqrt{a^2 + b^2})(x + y)$
$+ a^2 b^2 = 0.$

36. $x = a + r.$

37. $[4r^2 - 2(a - b)^2]^{\frac{1}{2}}.$

Exercise 23. Page 84.

1. $\frac{1}{2}\sqrt{29}$, $(1, -\frac{3}{2})$.
2. $\frac{1}{3}\sqrt{11}$, $(-\frac{4}{5}, \frac{2}{3})$.
3. $\frac{1}{2}\sqrt{34}$, $(\frac{3}{2}, \frac{5}{2})$.
4. b, $\left(\dfrac{b}{\sqrt{1+a^2}}, \dfrac{ab}{\sqrt{1+a^2}}\right)$.
5. $x^2 + y^2 = 81$.
6. $(x-7)^2 + y^2 = 9$.
7. $(x+2)^2 + (y-5)^2 = 100$.
8. $x^2 + y^2 - 2a(3x+4y) = 0$.
9. $x^2 + y^2 + 2b^2 + c^2 = 2[(b+c)x + (b-c)y]$.
10. $3ab(x^2+y^2) + 2ab(a^2+b^2) = (5a^2+2b^2)bx + (5b^2+2a^2)ay$.
11. $x^2 + y^2 - 5x - 12y = 0$.
12. $x^2 + y^2 - 14x - 4y - 5 = 0$.
13. $x^2 + y^2 + 14x + 14y + 49 = 0$.
14. $x^2 + y^2 \mp 2rx - 2ry + r^2 = 0$, $x^2 + y^2 \pm 2rx + 2ry + r^2 = 0$.
15. $x^2 + y^2 - 2ax - 2ay + a^2 - \dfrac{b^2}{4} = 0$.
16. $x^2 + y^2 = \frac{9}{5}$.
17. $5(x^2+y^2) - 10x + 30y + 49 = 0$.
18. $\begin{cases} (x-3)^2 + (y-1)^2 = 5, \\ (x+\frac{17}{6})^2 + (y+\frac{19}{6})^2 = \frac{5}{4}. \end{cases}$
19. $x^2 + y^2 - 30x - 52y = 0$.
20. $x^2 + y^2 + 50x + 88y - 50 = 0$.
21. $\begin{cases} x^2 + y^2 - 36x - 46y + 324 = 0, \\ 25x^2 + 25y^2 - 80x - 494y + 64 = 0. \end{cases}$
22. $(6, 2)$, 5.
23. -15.
24. -10.
25. $\frac{3}{2}\sqrt{26}$.
26. $\sqrt{10}$.
27. $x_1x + y_1y = x_1^2 + y_1^2$.
28. (i) $D^2 = 4AC$, (ii) $E^2 = 4AC$, (iii) $D^2 = E^2 = 4AC$.
29. $r^2 = 2rmc + c^2$.
30. $k = 40$, or -10.
31. $x^2 + y^2 = \pm ay\sqrt{2}$, or $= \pm ax\sqrt{2}$.
32. $x^2 + y^2 \pm 2a(x \pm y) = 0$.
33. $2(x^2 - ax + y^2 - r^2) + a^2 = 0$.
34. $x - y = 0$.
35. $4x + 3y = 0$.
36. $(18 \pm 2\sqrt{41})x - 5y = 0$.
37. $x + \sqrt{3}y \pm 20 = 0$.
38. $x + y - 10 = 0$.
40. $\frac{1}{2}(35 + 24\sqrt{30})$.
43. $135°$.
44. $(7, -5)$ and $(-6\frac{7}{29}, 9\frac{26}{29})$.
45. $\begin{cases} (x+4)^2 + (y+10)^2 = 85, \\ \left(x - \dfrac{514}{169}\right)^2 + \left(y + \dfrac{670}{169}\right)^2 = \dfrac{85}{169^2}. \end{cases}$
46. The circle $(x - x_1)^2 + (y - y_1)^2 = r^2$.
47. The circle $(x-a)^2 + (y-b)^2 = (r + r')^2$, or $(x-a)^2 + (y-b)^2 = (r - r')^2$.
48. The circle $(x-a)^2 + (y-b)^2 = r^2 + t^2$.

49. Take A as origin, and let the radius of the circle $= r$; then the locus is the circle $x^2 + y^2 = rx$.

50. Take A as origin, and let the radius of the circle $= r$; then the locus is the circle $x^2 + y^2 = \dfrac{2mrx}{m+n}$.

51. Take A as origin, AB as axis of x, and let $AB = a$; then the locus is the circle $(m^2 - n^2)(x^2 + y^2) - 2am^2x + a^2m^2 = 0$.

52. Take AB as the axis of x, the middle point of AB as origin, and let $AB = 2a$; then the locus is the circle $2(x^2 + y^2) = k^2 - 2a^2$.

53. Using the same notation as in No. 52, the locus is the straight line $4ax = \pm k^2$.

54. Taking the fixed lines as axes, the locus is the circle $4(x^2 + y^2) = d^2$.

55. Take the base as axis of x, its middle point as origin, and let the length of the base $= 2a$, and the constant angle at the vertex $= \theta$. Then the locus is the circle $x^2 + y^2 - 2a \cot\theta\, y = a^2$.

56. Take A as origin, AB as axis of x, and let $AB = a$, $AC = b$. Then the locus is the circle $(x - \frac{1}{2}a)^2 + y^2 = \dfrac{b^2}{4}$.

57. The circle $x^2 + y^2 = \dfrac{4r^4}{4r^2 - l^2}$, where l is the length of the chord.

58. The locus is a circle.

Exercise 24. Page 97.

1. $7x - 6y = 0$.

2. $x - y = 0$.

4. $x + y = r$, $2x + 3y = r$, $(a + b)x + (a - b)y = r^2$.

5. $13x + 2y = 49$.

6. The tangent at (h, k).

7. (i) $2x + 3y = 4$, (ii) $3x - y = 4$, (iii) $x - y = 4$.

8. (i) $(20, 30)$, (ii) $(21, -14)$, (iii) $(35a, 35b)$.

9. $(6, 8)$.

10. $\left(-\dfrac{Ar^2}{C}, -\dfrac{Br^2}{C}\right)$.

11. $(4, \pm 3)$, $4x \pm 3y = 25$.

16. $h^2 + k^2 - r^2$.

17. 3.

18. $12x + 17y - 51 = 0$.

19. $x + y - 2 = 0$.

20. $(a^2 - ab)x - (ab - b^2)y + ac = 0$.

21. $x - y = 0$, $\sqrt{\frac{1}{2}(a + b)^2 - 4c}$.

22. $(-2, -1)$.

Exercise 26. Page 109.

1. Writing $x+1$ for x, and $y-2$ for y, and reducing, we have $y^2=4x$.
2. $x^2+y^2=r^2$.
3. $x^2+y^2=2rx$.
4. $x^2+y^2=-2ry$.
5. $x^2+y^2=r^2$.
6. $2xy=a^2$.
7. $x^2-y^2+2=0$.
8. (i) $\rho=\pm a$, (ii) $\rho^2\cos 2\theta=a^2$.
9. (i) $\rho=4a\tan\theta\sec\theta$, (ii) $(a+\rho\cos\theta)^2=4a\rho\sin\theta$.
10. (i) $x^2+y^2=a^2$, (ii) $x^2+y^2=ax$, (iii) $x^2-y^2=a^2$.
11. $x+y=0$.
12. $2x-5y+10=0$.
13. $12x^2+16xy+4y^2+1=0$.
14. $x^2+y^2=25$.
15. $x^2-6xy+y^2=0$.
16. $xy=3$.
17. $y^2=2a(x\sqrt{2}-a)$.
18. $4xy=25$.

Exercise 27. Page 111.

1. $b\sqrt{3}$.
2. $4\sin\frac{1}{2}\omega$.
3. $\sqrt{13-12\cos\omega}$.
4. $\sqrt{a^2+b^2-2ab\cos(\theta-\phi)}$.
5. $2a\sin\theta$.
6. $2a\cos\theta$.
7. $a\sqrt{5-2\sqrt{3}}$.
9. $2x^2+2xy+y^2=1$.
10. $2x^2+y^2=6$.
11. $y=0$.
12. $9x^2+25y^2=225$.
13. $\rho=8a\cos\theta$.
14. $\rho=\pm 4a$.
15. $\rho^2\sin^2\theta-5\rho\cos\theta=\frac{25}{4}$.
16. $\rho^2=49\sec 2\theta$.
17. $\rho^2=k^2\cos 2\theta$.
18. $xy=a^2$.
19. $(x^2+y^2)^{\frac{3}{2}}=2kxy$.
20. $x^3-y^3+(3x-3y-5k)xy=0$.
21. $\tan^{-1}\frac{7}{5}$.
22. (i) $\tan^{-1}\left(-\frac{A}{B}\right)$, (ii) $\tan^{-1}\frac{B}{A}$.

Exercise 28. Page 117.

2. $y^2=4p(x-p)$.
3. $y^2=4p(x+p)$.
4. (i) $y^2=10x$, (ii) $y^2=10x+25$, (iii) $y^2=10x-25$.
5. (i) $y^2=16x$, (ii) $y^2=16x+64$, (iii) $y^2=16x-64$.

6. (0, 0), (2, 6).

7. 6, 15, $\frac{a}{b}$.

8. (4, 6) and (25, 15).

9. (12, 6).

10. The line $x = 9$ meets the parabola in (9, 6) and (9, -6). The line $x = 0$ passes through the vertex. The line $x = -2$ does not meet the parabola.

11. The line $y = 6$ meets the parabola in (9, 6). The line $y = -8$ meets the parabola in (16, -8).

12. $p = 4$.

13. (0, 0), (2, 8).

14. (i) $y = 0$, (ii) $x = -2$, (iii) $x = 2$, (iv) $4x \pm 3y - 8 = 0$, (v) $y = -2x$.

15. (i) $4x - 5y + 24 = 0$, (ii) $x^2 + y^2 - 20x = 0$.

16. $3p$.

17. $8p\sqrt{3}$.

24. The latus rectum of each $= 4p$. The common vertex is at the origin. The axis of x is the axis of (i) and (ii); that of y is the axis of (iii) and (iv). Parabola (i) lies wholly to the *right* of the origin, (ii) wholly to the *left*, (iii) wholly *above*, (iv) wholly *below*. We may name them as follows:

(i) is a right-handed X-parabola.
(ii) is a left-handed X-parabola.
(iii) is an upward Y-parabola.
(iv) is a downward Y-parabola.

Exercise 29. Page 121.

6. $x - 4y + 20 = 0$, $4x + y - 90 = 0$.

7. Tangents $\begin{cases} x - y + 3 = 0, \\ x + y + 3 = 0; \end{cases}$ normals $\begin{cases} x + y - 9 = 0, \\ x - y - 9 = 0. \end{cases}$

These lines enclose a square whose area $= 72$.

8. Tangent $= \sqrt{266}$, normal $= \sqrt{95}$, subtangent $= 14$, subnormal $= 5$.

9. (5, 10).

13. $\dfrac{p}{m\sqrt{1+m^2}}$, $\dfrac{-p}{m}\sqrt{m^2+1}$.

14. $[\sqrt{x_1 x_2}, \tfrac{1}{2}(y_1 + y_2)]$.

15. $x + y + p = 0$, point of contact $(p, -2p)$, intercept $= -p$.

16. Equations of the tangents $y\sqrt{3} = \pm x \pm 3p$, required point $(-3p, 0)$.

17. For the two points whose coördinates are

$$x = \frac{p}{8}(1 \pm \sqrt{17}),\ y = \pm p\sqrt{\frac{1 + \sqrt{17}}{2}}.$$

18. For the points (0, 0) and $(3p, \pm 2p\sqrt{3})$.

19. $9x - 6y + 5 = 0$, $(\frac{5}{9}, \frac{5}{3})$.

20. $x - 2y + 12 = 0$, (12, 12).
$4x + 2y + 3 = 0$, $(\frac{9}{4}, -3)$.

21. $y = x(\pm\sqrt{2} - 1) + 4(\pm\sqrt{2} + 1)$. **22.** $\dfrac{4\sqrt{p(p+x_1)^3}}{x_1}$.

24. By the secant method we find that the equation of the tangent at (x_1, y_1) is

$$\frac{y - y_1}{x - x_1} = \frac{4}{y_1 - 3}.$$

The points of contact are $(-1, 11)$ and $(-1, -5)$; hence the tangents are

$$x - 2y + 23 = 0$$

and

$$x + 2y + 11 = 0.$$

25. $\begin{cases} \text{(i) } y_1 y = -2p(x + x_1), \\ \text{(ii) } x_1 x = 2p(y + y_1), \\ \text{(iii) } x_1 x = -2p(y + y_1). \end{cases}$

Exercise 30. Page 123.

1. $y^2 = 24x - 144$.

2. $y^2 = 16x$.

3. $y^2 = -17x$.

4. $\begin{cases} 2y^2 - 11x + 12y + 73 = 0, \\ 2y^2 + 11x + 12y - 37 = 0. \end{cases}$

5. $(y + 7)^2 = 4(x - 3)$.

6. $3y^2 = 4x$.

7. $2x^2 = 9y$.

8. $\frac{3}{2}$, $8x + 3 = 0$, $8x \pm 15y - 3 = 0$.

10. $\begin{cases} 4 \text{ on } OX; \\ 8 \text{ and } -2 \text{ on } OY. \end{cases}$

11. $4(2 \pm \sqrt{3})p$.

12. $\begin{cases} \text{(i) } y = x + 2, \\ \text{(ii) } -2\sqrt{2}, \\ \text{(iii) } x + y - 6 = 0. \end{cases}$

13. $x + y - 6 = 0$.

14. $y - y_1 = \dfrac{2p}{y_1}(x - x_1)$.

15. $(8, 4)$, $(2, 10)$.

16. $(2, 4)$, $(11, 10)$.

17. $y = \dfrac{b \pm \sqrt{b^2 + 4ap}}{2a}x$.

18. $\begin{cases} \text{A left-handed } X\text{-parabola.} \\ \text{Latus rectum} = 2. \\ \text{Vertex, } (-2, 0). \\ \text{Focus, } (-\frac{5}{2}, 0). \\ \text{Directrix, } x = -\frac{3}{2}. \end{cases}$

19. $-2a\sqrt{2}$.

20. $y^2 = -9x$.

21. $y^2 = 8x$.

22. $y^2 = \dfrac{4r^2 - t^2}{r}x$.

23. $y^2 = \dfrac{n^2}{r}x$.

24. $y^2 = \dfrac{2n^2}{\sqrt{n^2 + t^2}}x$.

25. $y^2 = 2(2r - s)x$.

26. $4p\sqrt{2}$.

27. The equation of the circle is $(x - 3)^2 + (y - \frac{5}{2})^2 = \frac{25}{4}$.

28. $(-3p, 0)$.

29. $\left(\dfrac{p}{3}, \pm \frac{2}{3}p\sqrt{3}\right)$.

30. $\begin{cases} (p, \pm 2p); \\ 45^\circ \text{ and } 135^\circ. \end{cases}$

31. $4p^2$.

34. The parabola $y^2 = px$.

The loci in Nos. 35–38 are parabolas, the latus rectum in each being half that of the given parabola. If the given parabola is $y^2 = 4px$, the equations of the loci are:

35. $y^2 = 2px - p^2$.

36. $y^2 = 2px - 2p^2$.

37. $y^2 = 2px$.

38. $y^2 = 2px + 2p^2$.

39. The straight line $y = pk$.

40. The parabola $y^2 - 4px = p^2k^2$.

41. The straight line $kx = p$.

42. The circle $(x-p)^2 + y^2 = \dfrac{p^2}{k^2}$.

43. Take the given line as the axis of y, and a perpendicular through the given point as the axis of x, and let the distance from the point to the line $= a$. The locus is the parabola $y^2 = 2a\left(x - \dfrac{a}{2}\right)$.

Exercise 31. Page 134.

10. $3x - 5y - 6 = 0$.

11. $8y - 25 = 0$.

12. $13x + 22y + k = 0$.

13. $x - y - 1 = 0$.

14. $ay = 2px$. The chord is parallel to the tangent at the end of the diameter.

15. $y^2 = 52x$.

18. Writing the equation in the form $(y-3)^2 = 8(x-2)$, and passing to parallel axes through $(2, 3)$, we have $y^2 = 8x$. 8, $(2, 3)$, $(4, 3)$, $y = 3$, $x = 0$.

19. A numerically, $\left(\dfrac{B^2 - 4C}{4A},\ -\dfrac{B}{2}\right)$, $y = -\dfrac{B}{2}$.

20. B numerically, $\left(-\dfrac{A}{2},\ \dfrac{A^2 - 4C}{4B}\right)$, $x = -\dfrac{A}{2}$.

21. Take the given line as the axis of y, and a perpendicular through the centre of the given circle as the axis of x. Let the radius of the circle $= r$; distance from the centre to the given line $= a$. There are two cases to consider, since the circles may touch the given circle either externally or internally. The two loci are the parabolas

$$y^2 = 2(a \pm r)x + r^2 - a^2.$$

22. Let $2a$ be the given base, ab the given area; take the base as axis of x, its middle point as origin; then the locus is the parabola

$$x^2 + by = a^2.$$

Exercise 32. Page 144.

1. 5, 4, 3, $\frac{3}{5}$. **2.** $\sqrt{2}$, 1, 1, $\frac{1}{2}\sqrt{2}$. **3.** 2, $\sqrt{3}$, 1, $\frac{1}{2}$.

4. $\dfrac{1}{\sqrt{A}}$, $\dfrac{1}{\sqrt{B}}$, $\sqrt{\dfrac{B-A}{AB}}$, $\sqrt{\dfrac{B-A}{B}}$, when $A < B$.

5. $\frac{6}{7}\sqrt{6}$.
6. $e = \frac{1}{2}\sqrt{3}$.
7. $4x^2 + 9y^2 = 144$.
8. $25x^2 + 169y^2 = 4225$.
9. $16x^2 + 25y^2 = 3600$.
10. $16x^2 + 25y^2 = 1600$.
11. $25x^2 + 169y^2 = 4225$.
12. $3x^2 + 7y^2 = 115$.
13. $x^2 + 2y^2 = 100$.
14. $8x^2 + 9y^2 = 8a^2$.
16. $2 : \sqrt{3}$.
17. $x = y = \pm \frac{ab}{\sqrt{a^2 + b^2}}$.
18. $(\frac{1}{3}, \frac{4}{3})$, $(-\frac{5}{3}, -\frac{2}{3})$.
19. $(1, 2)$, $(1, -2)$.
20. $(3, 1), (3, -1), (-3, 1), (-3, -1)$.
21. The equation of the locus is $x^2 + 4y^2 = r^2$.

22. Taking as axes the two fixed lines, and putting $AP = a$, $BP = b$, the acute angle between AB and the axis of $x = \phi$, we find that

$$x = a \cos\phi, \; y = b \sin\phi.$$

Therefore P describes an ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

23. The two straight lines $y = \pm x\sqrt{-\frac{A}{B}}$. The locus is imaginary when y is imaginary; that is, when A and B have like signs.

29. The equations of the sides are

$$x = \pm \frac{ab}{\sqrt{a^2+b^2}}, \quad y = \pm \frac{ab}{\sqrt{a^2+b^2}};$$

$$\text{area} = \frac{4a^2b^2}{a^2+b^2}.$$

Exercise 33. Page 151.

1. $\begin{cases} 4x \pm 9y = 35, \\ 9x \mp 4y = 6. \end{cases}$
2. $\begin{cases} 2x \mp 3y\sqrt{3} + 12 = 0, \\ 6x\sqrt{3} \mp 4y + 5\sqrt{3} = 0. \end{cases}$
3. $\begin{cases} x + 4y = 10, \\ 4x - y - 6 = 0; \\ -8, -\frac{1}{2}. \end{cases}$
5. $\frac{a^2}{m^2} + \frac{b^2}{n^2} = 1$.
6. $9x^2 + 25y^2 = 225$.
7. $2y = x \pm 10$.
8. $4x - 3y \pm \sqrt{107} = 0$.
9. $x = \pm \frac{a^2}{\sqrt{a^2+b^2}}, y = \pm \frac{b^2}{\sqrt{a^2+b^2}}$.
10. Same answers as No. 9.
11. $b^2 : a^2$.
12. $x = \pm \frac{a}{\sqrt{2}}, \quad y = \pm \frac{b}{\sqrt{2}}$.
13 $y = 4$, $3x + 2y = 17$.

14. The equation $\pm\sqrt{5}x \pm 3y = 9a$ represents the four tangents.

15. $a\sqrt{1-e^2\cos^2\phi}$.

16. $\frac{1}{2}(a^2\csc\phi\sec\phi - c^2\cot\phi)$.

17. The extremities of the latera recta.

19. The method of solving this question is similar to that employed in § 136. The required locus is the auxiliary circle $x^2+y^2=a^2$.

Exercise 34. Page 152.

1. $\begin{cases} x=8,\ 40y=9x+72; \\ \frac{18}{5},\ \frac{32}{5}. \end{cases}$

2. Within.

3. $\frac{1}{3}\sqrt{3}$.

4. $\frac{1}{2}\sqrt{2}$.

5. $\dfrac{2\sqrt{3}}{1+\sqrt{13}}$.

6. $x+y=\pm\sqrt{a^2+b^2}$.

7. $bx+cy\mp b\sqrt{a^2+c^2}=0$.

8. $bx+ay\mp ab\sqrt{2}=0$.

9. $\dfrac{x}{a}\cos\phi+\dfrac{y}{b}\sin\phi=1$.

10. $\dfrac{ab}{\sqrt{a^2-e^2x_1^{\ 2}}}$.

11. $a\sqrt{1-e^2\cos^2\phi}$.

12. $c^2:b^2$.

13. $a^2-e^4x_1^{\ 2}$.

14. $\sqrt{(1-e^2)(a^2-e^2x_1^{\ 2})}$.

15. $\tan\phi=\dfrac{\sqrt{1-e^2}}{e}=\dfrac{b}{c}$.

18. The locus is the minor axis produced.

19. The ellipse $4\left(x-\dfrac{a}{2}\right)^2+y^2=r^2$; centre is $\left(\dfrac{a}{2},\ 0\right)$; semi-axes are $\dfrac{r}{2}$ and r.

20. The ellipse $a^2\left(y-\dfrac{b}{2}\right)^2+b^2x^2=\dfrac{a^2b^2}{4}$; centre is $\left(0,\ \dfrac{b}{2}\right)$; semi-axes are $\dfrac{a}{2}$ and $\dfrac{b}{2}$.

In 21–23 take the base of the triangle as the axis of x, and the origin at its middle point.

21. The ellipse $(s^2-c^2)x^2+s^2y^2=s^2(s^2-c^2)$.

22. The ellipse $kx^2+y^2=kc^2$.

23. The circle $(x+c)^2+y^2=4a^2$.

Exercise 35. Page 164.

1. 8π.

2. $\frac{8}{3}\sqrt{3}$.

3. $20x+63y-36=0$.

5. $\left(-\dfrac{Aa^2}{C},\ -\dfrac{Bb^2}{C}\right)$.

8. (i) $m_1^{\ 2}=\dfrac{b^2}{a^2}$, (ii) $m_1^{\ 2}=\dfrac{b}{a}$, (iii) $m_1^{\ 2}=1$.

9. $a = l\sqrt{\dfrac{1 - e^2\cos^2\theta}{1 - e^2}}$, $b = l\sqrt{1 - e^2\cos^2\theta}$.

11. $3x + 8y = 4$, $2x - 3y = 0$.

12. Area $= \dfrac{b^2}{2a}(m + n)$, m and n being the two segments (use the polar equation).

13. $26x + 33y - 92 = 0$.

14. $x + 2y = 8$.

15. $b^2x + a^2y = 0$, $b^2x - a^2y = 0$, $a^3y + b^3x = 0$, $bx + ay = 0$.

17. $a^2y_1x = b^2x_1y$.

23. $\dfrac{x}{a} \mp \dfrac{y}{b} = 0$.

24. $bx\sqrt{l^2 - b^2} \pm ay\sqrt{a^2 - l^2} = 0$.

25. $e = \frac{1}{3}\sqrt{6}$.

26. See § 148.

29. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{2x}{a}$.

30. $\rho = \dfrac{a(1 - e^2)}{1 + e\cos\theta}$.

31. $\rho^2 = \dfrac{b^2}{1 - e^2\cos^2\theta}$.

32. $16x^2 + 49y^2 - 128x - 686y + 1873 = 0$.

33. $2a = 18$, $2b = 10$.

34. $\dfrac{25x^2}{144} + y^2 = 5x$.

35. $\cos\phi = \sqrt{\dfrac{c^2 - b^2}{a^2 - b^2}}$.

36. $\tan\theta = \dfrac{a - b}{\pm\sqrt{ab}}$.

37. Find the ratio of y_1 to the intercept on the axis of y.

38. $b^2hx + a^2ky = b^2h^2 + a^2k^2$.

41. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2$.

42. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \frac{1}{2}$.

43. The ellipse $b^2x^2 + a^2y^2 = b^2c^2$.

44. $(-1, 1)$, $a = 2$, $b = 1$.

45. $\sqrt{\dfrac{K}{A}}$, $\sqrt{\dfrac{K}{B}}$, in which $K = -F + \dfrac{D^2}{4A} + \dfrac{E^2}{4B}$.

46. The ellipse $25x^2 + 16y^2 - 48y = 64$.

Exercise 36. Page 174.

1. $\dfrac{x^2}{64} - \dfrac{y^2}{49} = 1$.

2. $\dfrac{4x^2}{25} - \dfrac{y^2}{36} = 1$.

3. $3x^2 - y^2 = 3a^2$.

4. $625x^2 - 84y^2 = 10{,}000$.

5. $2x^2 - 2y^2 = c^2$.

7. $a = 4$, $b = 3$, $c = 5$, $e = \frac{5}{4}$, latus rectum $= \frac{9}{2}$.

8. $16y^2 - 9x^2 = 144$, transverse axis $= 6$, conjugate axis $= 8$, distance between foci $= 10$, latus rectum $= \frac{32}{3}$.

9. $a : b = 1 : \sqrt{3}$. **11.** $e = \sqrt{2}$. **12.** $(5, -6\frac{2}{3})$.

14. Foci, $(5, 0)$, $(-5, 0)$; asymptotes, $y = \pm \frac{4}{3}x$. **17.** b.

Exercise 37. Page 176.

1. $16x - 9y = 28$, $9x + 16y = 100$; $\frac{9}{4}$, $\frac{64}{9}$.

3. $x^2 - y^2 = 9$, $(5, 4)$.

4. The four points represented by

$$x = \frac{\pm a^2}{\sqrt{a^2 - b^2}}, \quad y = \frac{\pm b^2}{\sqrt{a^2 - b^2}}.$$

9. $\frac{a}{\sqrt{3}}$.

10. $\frac{a^2}{m^2} - \frac{b^2}{n^2} = 1$.

11. When a is less than b.

12. The circle $x^2 + y^2 = a^2$.

Exercise 38. Page 177.

1. $2be$, ae^2.

2. 14 and 6; $(-8, \pm 3\sqrt{3})$.

3. The sum $= 2ex$.

8. $(a, b\sqrt{2})$, $(a, -b\sqrt{2})$.

10. They are equal.

11. $y = x\sqrt{2} + a$.

12. $(0, \pm\sqrt{a^2 - b^2})$.

13. $b^2 > a^2$.

14. $64x - 9y - 741 = 0$.

15. $y = 4x \pm 8\sqrt{2}$.

16. $\frac{a^2b^2}{a^2 + b^2}$.

Exercise 39. Page 188.

1. $9x + 12y + 16 = 0$.

2. $x = \pm\frac{a}{e}$.

3. $\frac{\pi}{2}$.

4. $\left(-\frac{Aa^2}{C}, \frac{Bb^2}{C}\right)$.

5. $x + a = 0$.

6. a.

8. $75x - 16y = 0$.

9. $245x - 12y - 1189 = 0$.

10. $\frac{5}{3}\sqrt{3}$.

17. See § 140.

18. $\frac{x}{x_1} + \frac{y}{y_1} = 2$.

19. (i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{2x}{a} = 0.$ (ii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{2x}{a} = 0.$

20. $\rho = \frac{a(e^2 - 1)}{1 - e\cos\theta}.$

21. $\rho^2 = \frac{b^2}{e^2\cos^2\theta - 1}.$

23. The hyperbola $3x^2 - y^2 + 20x - 100 = 0$. The centre is the point $(-\frac{10}{3}, 0)$. Changing the origin to the centre, we obtain $9x^2 - 3y^2 = 400$.

24. Writing the equation in the form $(x-1)^2 - 4(y+2)^2 = 4$, and changing the origin to $(1, -2)$ we obtain $\frac{x^2}{4} - \frac{y^2}{1} = 1$. The centre is $(1, -2)$, $a = 2$, $b = 1$.

25. Centre is $\left(\frac{-D}{2A}, \frac{-E}{2B}\right)$, semi-axes are $\sqrt{\frac{K}{A}}$, $\sqrt{\frac{K}{B}}$, in which $K = -F + \frac{D^2}{4A} + \frac{E^2}{4B}$.

26. The locus is the curve $2xy - 7x + 4y = 0$. If we change the origin to the point (h, k), we can so choose the values of h and k as to eliminate the terms containing x and y. Making the change, we obtain

$$2xy + (2k - 7)x + (2h + 4)y - 7h + 4k + 2hk = 0.$$

If we choose h and k so that $2h + 4 = 0$, and $2k - 7 = 0$, that is, if we take $h = -2$, $k = \frac{7}{2}$, the terms containing x and y vanish, and the equation becomes $xy = -7$. Hence we see (§ 182, Cor.) that the locus is an equilateral hyperbola, whose branches lie in the second and fourth quadrants, and that the new axes of coördinates are the asymptotes.

27. The equilateral hyperbola $2xy = a^2$.

28. Taking the base as axis of x, and the vertex of the smaller angle as origin, the locus consists of the axis of x and the hyperbola $3x^2 - y^2 - 2ax = 0$.

Exercise 40. Page 206.

1. The ellipse $72x^2 + 48y^2 = 35$.

2. The ellipse $4x^2 + 2y^2 = 1$.

3. The hyperbola $32x^2 - 48y^2 = 9$.

4. The ellipse $9x^2 + 3y^2 = 32$.

5. The hyperbola $4x^2 - 4y^2 + 1 = 0$.

6. The parabola $y^2 = -\frac{2}{3}x$.

7. The parabola $y^2 = 2x\sqrt{2}$.

8. The parabola $y^2 = 3x\sqrt{2}$.

9. The parabola $y^2 = 2x$.

10. The ellipse $4x^2 + 9y^2 = 36$.

11. The point $(0, 0)$.

12. The hyperbola $4x^2 - 9y^2 = 36$.

13. The straight lines $y = x$, $y = -5$.

Equation for any straight line.